Quantum Entanglement in typical and Not so typical States

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Correlations and coherence in quantum systems
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Quantum Entanglement: problem or opportunity?

"old" **problem:** non locality/non realism, ***EPR*** paradox, Bell Inequality violations,…..*Quantum Weirdness*

"new" **opportunity:** quantum computation/communication speed-ups, quantum simulations, understanding origin of stat-mech,….*Harnessing Quantum Weirdness*

Q: how much entangled is a Typical Quantum State?

A:Well, it depends on what we mean by Typical…..

Let's see…..

Typical 1: Random Quantum State (Haar distributed)

A1: (by far) most of the states are nearly maximally entangled

$$\Pr\left\{\Psi : Ent(\Psi)/Ent_{MAX} < 1-\alpha\right\} = O(e^{-\alpha N_A})$$

$\alpha > 0, N =$ **#** of particles in the subsystem A $\quad Ent_{TYP_1} \sim N_A$

Volume Law for Entanglement, extensive behavior, fulfilled by the Overwhelming majority of states

PROBLEM: ***Typ1*** states are exponentially hard to prepare
=sampling the Haar Distr requires exp depth stochastic quantum circuits

Hilbert Space is a "*convenient illusion*" (sic!)
D. Poulin et al, Phys. Rev. Lett. 106, 170501 (2011)

Typical 2: most of the physical states around us
Low energy states of local Hamiltonians,

A2: "area laws" are fulfilled (out of criticality)

1- gapped GS implies finite correlation length
2- just particles within a distance ξ from ∂A contribute ξ
to ***EE***

$$Ent_{TYP_2} \sim |\partial A|$$

Violations @criticality: $\xi \rightarrow \infty$

Example *1D spin chain: off-critical ➔* ***EE*** *saturation*
Critical ➔logarithmic divergence with size (Rico et al PRL 2003)

Consequences
"Good": ***EE*** is a powerful indicator of Quantum Phase Transitions
"Bad": critical systems are hard to simulate

Rigorous proofs: typically (very) hard….
Q: can we make it any easier??

Lorenzo CV and I tried….
Entanglement Susceptibility
arXiv:1205.2507

Typical 3: Stochastic Local Quantum Circuits

Physical Model: limited control resources
a) spatial resolution; b) Hamiltonian control

Mathematical Model: at each tick of the clock one enacts a Random Unitary acting on a small subset of vertices

System: collection o f |*V*| local *d*-dimensional sub-systems e.g., qubits

Global state space: $$H_V := \otimes_{i\in V} h_i \cong (C^d)^{\otimes |V|}$$

V={i}= set of vertices; |*V*|= size of *V=O(N)* [*N* is large **TL**]

We are going to consider random variables valued in the unitary group Of *A(V)*: $\Omega \subseteq V \Rightarrow H_\Omega =: \otimes_{i\in\Omega} h_i;\qquad A(V) =: L(H_\Omega)\otimes Id_{V\setminus\Omega}$

Region — Region state-space — Region operator algebra

PREPARE $\rho_{in} := |\Psi_{in}\rangle\langle\Psi_{in}|; \qquad /* \; |\Psi_{in}\rangle \in H_V \; */$

```
for(i=1;I<=k; i++){                    /* time iterations. Total time=k */
    SELECT region  Ω_i ⊆ V;            /* according q^(i) */
    ENACT unitary  U_i ∈ A(Ω_i);       /* according dμ^(i)(U | Ω_i) */
}
```

$q^{(i)} : \Omega \subseteq V \rightarrow q^{(i)}(\Omega) \in [0,1]$ Probability distributions over the regions of *V*

$d\mu^{(i)}(U \,|\, \Omega_i)$ Probability densities over the unitary part of $A(\Omega_i)$; e.g., Haar

Locality assumption: the $q^{(i)}$ are supported of small regions $|\Omega_i| = O(1)$ i.e., U_i $is\,|\Omega_i|-local$

The Circuit: $\mathbf{C}(\rho_{in}) := U_k U_{k-1} \cdots U_1 \rho_{in} U^*_1 \cdots U^*_{k-1} U^*_k$

$\mathbf{C} = \mathbf{C}[\{q^{(i)}\}_{i=1}^k, \{d\mu^{(i)}(\bullet \,|\, \Omega_i)\}_{i=1}^k]$ Random *CP*-map of *A(V)* into itself

$\mathbf{C}_k = \mathbf{C}_k[q, d\mu] =$ {circuits of length with schedule [q, $d\mu$] }

Model 1: Random Edge Model (**REM**)

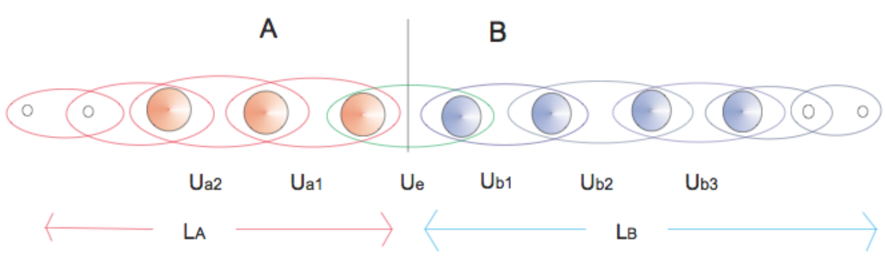
***G**=(V,E)* graph (*V*=vertices, *E*=edges)

$$q^{(i)} = q : \Omega \to \frac{1}{|E|} \sum_{e \in E} \delta(\Omega, e)$$

Uniform distribution over the edges of ***G***

$d\mu^{(i)}(\bullet \mid \Omega) =$ Haar distribution over the unitary part of $A(\Omega)$

Model 2: One dimensional architecture (**CHAIN**)



a) **Schedule** 1: *expanding*

$$(U_{a_{L-1}} U_{a_{L-2}} \cdots U_{a_1})(U_{b_{L-1}} U_{b_{L-2}} \cdots U_{b_1}) U_e$$

b) **Schedule** 2: *contracting*

$$U_e (U_{a_1} U_{a_2} \cdots U_{a_L})(U_{b_1} U_{b_2} \cdots U_{b_L})$$

$d\mu^{(i)}(\bullet \mid e_i) =$ Haar distribution over edge unitaries;
basic block *k=2L-1 (iters)*

Observables and Goal Functionals

$$F_O[\mathbf{C}] = Tr\left[O\mathbf{C}^{\otimes n}(\rho_{in})\right] = \langle \rho_{in}, \mathbf{C}_*^{\otimes n}(O)\rangle \qquad (O \in A(V))$$

Non linear functionals; Rand Var $F_O : \mathbf{C}_k \to \mathbf{R}$

Expectations: $\overline{F_O[\mathbf{C}]}^{\mathbf{C}_k} := \sum_{\mathbf{C}} p(\mathbf{C}) F_O[\mathbf{C}] = \langle \rho_{in}, \sum_{\mathbf{C}} p(\mathbf{C}) \mathbf{C}_*^{\otimes n}(O)\rangle$

Example: Local Purity $\| \rho_A \|_2^2 := Tr[\rho_A{}^2] = Tr[T_A \rho \otimes \rho]$

$T_A : H_V{}^{\otimes 2} \to H_V{}^{\otimes 2}$ Swap of the *A* factors

Generalization: Schatten Norms $\| \rho_A \|_p^p := Tr[\rho_A{}^p] = Tr[T^{(p)}{}_A \rho^{\otimes p}]$

$T^{(p)}{}_A : H_V{}^{\otimes p} \to H_V{}^{\otimes p}$ Cyclic shift of the *A* factors

Proxies of **Entanglement Entropy** $S(\rho_A) = -\lim_{p\to 1} \frac{1}{p-1} \log \| \rho_A \|_p^p$

Circuit super-operators

$$\overline{F_O[\mathbf{C}]}^{\mathbf{C}_k} := \langle \rho_{in}, R(O)\rangle \qquad R := \sum_{\mathbf{C}\in\mathbf{C}_k} p(\mathbf{C})\mathbf{C}_*^{\otimes n} = R^{(k)}R^{(k-1)}\cdots R^{(1)} \in \mathbf{CP}$$

$$R^{(i)}(O) := \sum_{\Omega\subseteq V} q^{(i)}(\Omega)\int d\mu^{(i)}(U|\Omega)U^{\otimes n}OU^{*\otimes n} \qquad (O\in A(V))$$

Goal functionals are matrix elements of the *R*'s

REM: $R = (R^{(1)})^k \qquad R^{(1)}(O) := \frac{1}{|E|}\sum_{e\in E}\int d\mu(U|e)U^{\otimes n}OU^{*\otimes n} =: \frac{1}{|E|}\sum_{e\in E} R_e(O)$

R_e Edge ***CP***-maps: **projections** on the commutant of the representation $U\in U(d^2)\to U^{\otimes n}\in L(H_e^{\otimes n})$ linear span of perms of perms $\mathbf{S}_n$

CHAIN: $R_{\exp} = (R_{b_{L-1}}R_{b_{L-2}}\cdots R_{b_1})(R_{a_{L-1}}R_{a_{L-2}}\cdots R_{a_1})R_e$ **(Schedule 1):**

$R_{comp} = R_e(R_{a_1}R_{a_2}\cdots R_{a_{L-1}})(R_{b_1}R_{b_2}\cdots R_{b_{L-1}})$ **(Schedule 2):**

Remark: Strings of non-commuting projections

Purity Dynamics and permutation algebra

$$R_e(T_A) = \chi_{E\setminus\partial A}(e)T_A + \chi_{\partial A}(e)N_d(T_{A\cup e} + T_{A\setminus e})$$ $[N_d := d(d^2+1)^{-1}]$

Key algebraic relation: the set of swaps is invariant (and reducible)

R is (non-negative) matrix on a $2^{|V|}-\dim$ space*:*
purity dynamics is a dynamical system on |V| qubits **!**

REM:

$$P_k := \langle \rho_{in}^{\otimes 2}, (R^{(1)})^k(T_A)\rangle \approx \left(1 - e_p \frac{|\partial A|}{|E|}\right)^k$$

$\rho_{in} = \otimes_{i\in V} |\phi_i\rangle\langle\phi_i|^{\otimes 2};$ $\quad k, \deg(i) = o(|\partial A|);$

$$e_p(d) = \frac{(d-1)^2}{d^2+1} = 1 - 2N_d$$ *Average "Entangling power" in d-dim*

PZ et al, Phys. Rev. A **62**, 030301 (2000)

$$S_k \geq -\log P_k \approx k e_p \frac{|\partial A|}{|E|}$$

Area Law (small k)
And linear in k **!**

CHAIN: use expanding schedule (k=# of blocks) $P_k \approx 2\left(\frac{1-e_p}{1+e_p}\right)^k$ $1 << k \le |A|$

Exponential decrease of average purity; 2-Renyi entropy increase

$\overline{S_2(k)} \ge -\log P_k \approx k \log d - \log 2$ **For *k=O(|A|)* we have a volume law**

Asymptotic purity: $P_\infty = \frac{d^{|A|} + d^{|B|}}{d^{|A|} d^{|B|} + 1} \le P_{\min} + d^{-|B|}$

(easy) bound:

$$D_k := \langle \| \mathbf{C}_k(\rho_{in}) - \mathbf{1}/d^{|A|} \|_1 \rangle_{\mathbf{C}_k} \le \sqrt{d^{|A|}} \sqrt{\frac{1}{d^{|B|}} + |P_k - P_\infty|}$$

$$|A| = (1-\alpha)|B| \quad AND \quad |P_k - P_\infty| = O(d^{-(1+\alpha)|A|})$$

$$0 \le \alpha \le 1, \qquad \Rightarrow D_k = O(d^{-\alpha|A|/2})$$

Markov inequality **==>** Vast majority of (*A*-reduced) states are exp close to the maximally mixed one

(critical) Question: How about convergence time T_{conv}?

Answer: study the spectral properties of *R*

$$sp(R) = \{\lambda_1 = 1 \geq \lambda_2 \geq \ldots \geq \lambda_{2^L}\}, \qquad |\lambda_i| \leq 1; \qquad \| R \|_\infty = 1$$

REM on a complete graph: Symmetry **==>** Exp complexity reduction: need to study **R** just in the *(L+1)-dim* symmetric subspace *(L=#nodes)!*

$$R \to \mathbf{R} = 1 - \frac{1}{|E|}\left[(L/2)^2 - (S^z)^2 + C_d\left(S^+\left(\frac{L}{2} + S^z\right) + S^-\left(\frac{L}{2} - S^z\right)\right)\right] =: 1 - \widehat{\mathbf{R}}$$

$$|A| = L/2 \qquad k >> T^{\varepsilon}_{Conv} =: \frac{C(L) + \log(1/\varepsilon)}{\log(1/\lambda_2(L))} \Rightarrow |P(k) - P(\infty)| \leq \varepsilon$$

$$C(L) \sim L, \qquad \lambda_2 = 1 - O(1/L) \Rightarrow \qquad T^{\varepsilon}_{Conv} = O(L^2 + L\log 1/\varepsilon)$$

Convergence time is polymonial in the system size!

A. Hamma, S. Santra and PZ, arXiv:1204.0288 **(PRA soon)**

Conclusions

0- Volume laws for entanglement are typical for random quantum states, boundary laws are typical for physical states

1- We introduced ***loc-RQC*** to generate physical states with a well defined probability distribution

2- Boundary laws are obtained for $k=O(1)$; volume for $k=O(|A|)$

3- Convergence time to (nearly)maximally mixed states is poly(|A|)

Do you wanna know a bit more?

A. Hamma, S. Santra and PZ, **Phys. Rev. Lett. 109, 040502 (2012)**

REM: The Complete Graph

$$R(T_A) = q(A)T_A + \frac{C_d}{|E|}\left\{N_B \sum_{i\in A} T_{A\setminus i} + N_A \sum_{j\in B} T_{A\cup j}\right\}, \qquad |E| = \binom{|V|}{2}; \qquad q(A) = \frac{1}{|E|}\left(\binom{|A|}{2} + \binom{|B|}{2}\right)$$

$$V = A \cup B$$

Mapping sets/qubits $\quad T_A \to \otimes_{i\in V} |\chi_A(i)\rangle \in (\mathbf{C}^2)^{\otimes|V|}$

$$S^\alpha = 1/2 \sum_{i\in V} \sigma_i^\alpha, \qquad (\alpha = x, y, z)$$

$$N_A \to |V|/2 + S^z$$

$$R \to \mathbf{R} = 1 - \frac{1}{|E|}\left[|V/2|^2 - (S^z)^2 + C_d\left(S^+\left(\frac{|V|}{2} + S^z\right) + S^-\left(\frac{|V|}{2} - S^z\right)\right)\right] =: 1 - \widehat{\mathbf{R}}$$

$$\|\widehat{\mathbf{R}}\| \le \frac{|V|^2}{2|E|}(1 + 2C_d) = \frac{|V|^2}{2|V|(2|V|-1)}\frac{(d+1)^2}{d^2+1} \le 1 \Longrightarrow sp(\mathbf{R}) \subset [0,1]$$

R has positive spectrum with max eigenvalue 1

$$P_k = \langle \Omega^{(+)}, R^k(T_A)\rangle = \langle \Omega^{(+)}, \sum_{B\subset V} (R^k)_{B,A} T_B\rangle = \sum_{B\subset V} (R^k)_{B,A}\langle \Omega^{(+)}, T_B\rangle =$$

$$\sum_{B\subset V} (R^k)_{B,A} = \langle \Phi | \mathbf{R}^k | A\rangle = \langle \Phi | P_{Symm} \mathbf{R}^k P_{Symm} | A\rangle \qquad |\Phi\rangle := (|0\rangle + |1\rangle)^{\otimes|V|} \in H_{Symm}$$

NB: We used $[P_{Symm}, \mathbf{R}^k] = 0$

We need to diagonalize **R** just in H_{symm}
A |V|+*1-dim* problem…..**!**