Quantum integrability in systems with finite number of levels

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In this talk (preview):

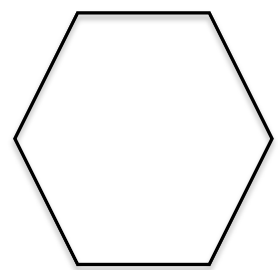
Q: What is quantum integrability? How is it defined?
Hamiltonian operator H is said to be integrable if…???

No clear unambiguous definition!
(See e.g. B. Sutherland, Beautiful Models (2004) for review)

e.g. no natural notion of an integral of motion: for any H_0 can find a full set of H_k such that $[H_0, H_k]=0$

$$H_0 = \sum_1^N E_n |n\rangle\langle n|, \quad H_k = |k\rangle\langle k|$$

1. Properties of quantum integrable models: Hubbard model on a ring
   - ✓ Exact solution via Bethe's Ansatz
   - ✓ Energy level crossings in violation of Wigner-v. Neumann noncrossing rule
   - ✓ Poisson level statistics



In this talk (preview):

2. Dynamical properties of integrable models
   - ✓ Exactly solvable multi-state Landau-Zener problems
   - ✓ Generalized Gibbs distribution
3. What is required of a good definition of quantum integrability?
4. Classical integrability
5. Difficulties in defining quantum integrability
6. Proposed definition – fix parameter dependence

$$H_i(u) = T_i + uV_i, \quad [H_i(u), H_j(u)] = 0 \quad \text{for all } u$$

7. Classification (types 1, 2,…) and explicit construction. Examples: Hubbard, XXZ, Gaudin models
8. Consequences: exact solution, level xings, Yang-Baxter, Poisson statistics
9. Type 1 = Gaudin magnets
10. Hubbard model: additional conservation laws, simplification of BA
11. Can we solve multi-state Landau-Zener for a much wider class of Hamiltonians?

Properties of quantum integrable models: Exact Solution

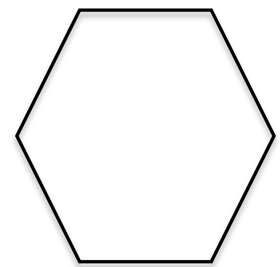
Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^\dagger_{j\,s} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{j\,s}) + U \sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***

tight-binding + onsite interactions, electrons on a ring

N=6 cites, 3 spin-up, ***M=3*** spin-down



Exact Solution (Bethe's Ansatz):

E.H. Lieb and F.Y.Wu (1969)

$$e^{6ik_j} = \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \sin k_j - iu/4}{\Lambda_\alpha - \sin k_j + iu/4}, \quad \prod_{\alpha=1}^{3} \frac{\Lambda_\alpha - \Lambda_\beta + iu/2}{\Lambda_\alpha - \Lambda_\beta + iu/2} = -\prod_{j=1}^{6} \frac{\Lambda_\beta - \sin k_j - iu/4}{\Lambda_\beta - \sin k_j - iu/4}$$

9 coupled nonlinear equations

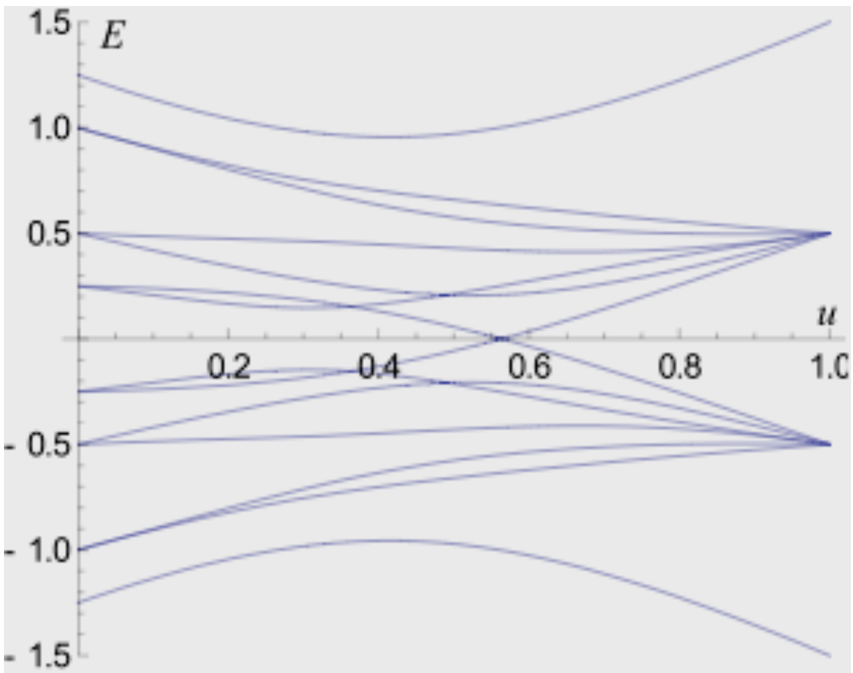
$$E = -\sum_{j=1}^{6} 2\cos k_j, \quad P = \sum_{j=1}^{6} k_j, \quad |P, S, S_z, \dots\rangle = \dots$$

Properties of quantum integrable models: Level crossings

Example: Hubbard model

$$\hat{H} = T \sum_{j,s=\uparrow\downarrow} (c^\dagger_{j\,s} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{j\,s}) + U \sum_j \hat{n}_{j\uparrow}\hat{n}_{j\downarrow}$$

H depends linearly on one parameter ***u=U/T***



Energies for a 14 x 14 block of 1d Hubbard on six sites characterized by a complete set of quantum numbers

H(u)=A+uB is a **14 x 14** Hermitian matrix linear in real parameter ***u***

"The noncrossing rule is apparently violated in the case of the 1d Hubbard Hamiltonian for benzene molecule *[six sites]*..."

Heilmann and Lieb (1971)

Commuting integrals (conservation laws)
Example: Hubbard model

$$\hat{H} \equiv \hat{H}_0(u) = \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{js} c_{j+1\,s} + c^\dagger_{j+1\,s} c_{js}) + u \sum_{j=1}^{N} \hat{n}_{j\uparrow} \hat{n}_{j\downarrow}$$

$$\hat{n}_{j\sigma} = c^\dagger_{js} c_{j\,s}$$

$$\hat{H}_1(u) = -i \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+2s} c_{js} - c^\dagger_{js} c_{j+2s}) - iu \sum_{j=1}^{N} \sum_{s=\uparrow\downarrow} (c^\dagger_{j+1s} c_{js} - c^\dagger_{js} c_{j+1s})(\hat{n}_{j+1,-s} + \hat{n}_{j,-s} - 1)$$

$$[\hat{H}_0(u), \hat{H}_1(u)] = 0 \quad \text{for all } u$$

B. S. Shastry, PRL (1986)

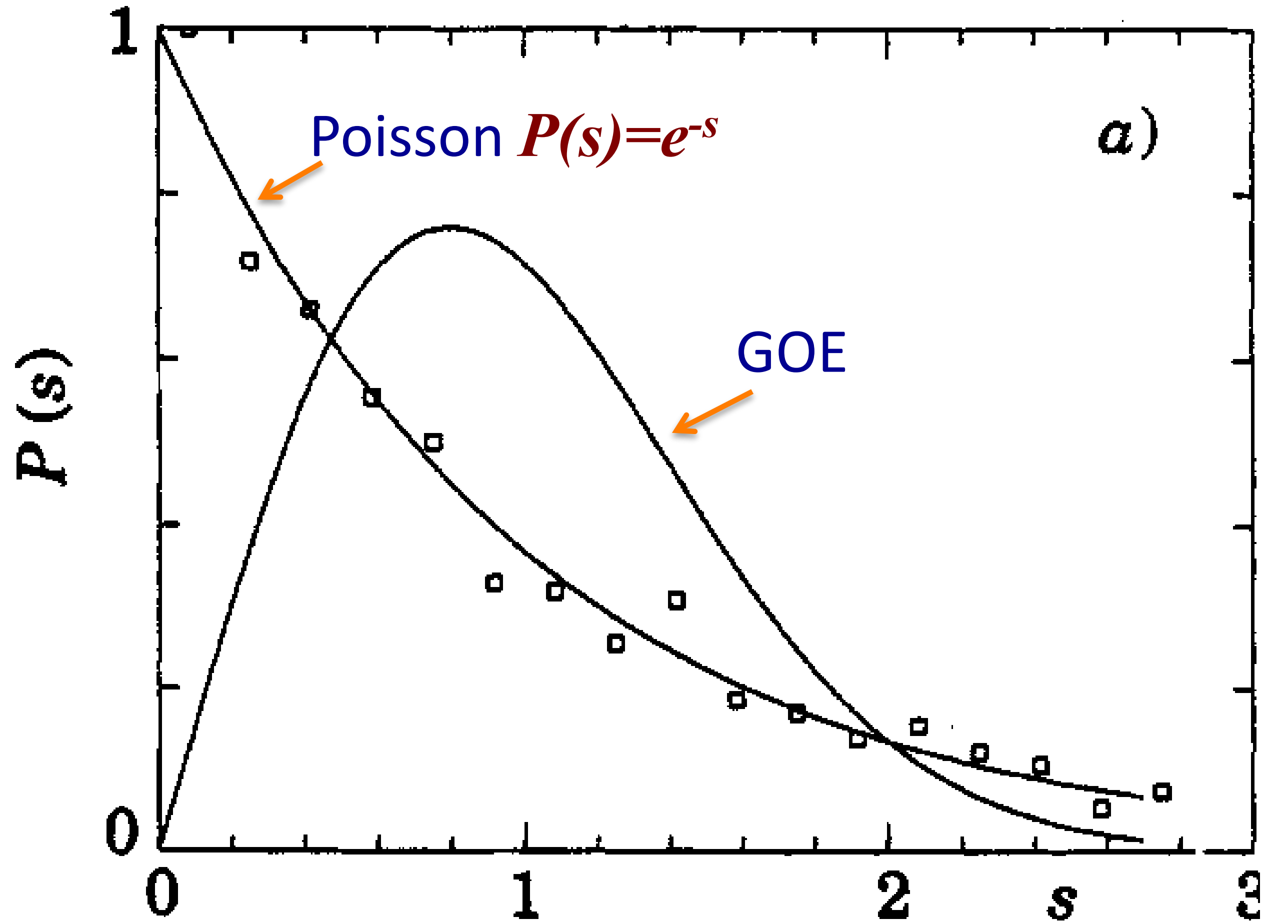
Both the Hamiltonian and first conserved current are Hermitian matrices linear in real parameter *u*

$H_2(u)$, $H_3(u)$, $H_4(u)$,… **- in principle, infinitely many integrals of motion can be found from Shastry's transfer matrix (but not all of them are nontrivial for finite *N*)**

Properties of quantum integrable models: Poisson statistics

Example: Hubbard model

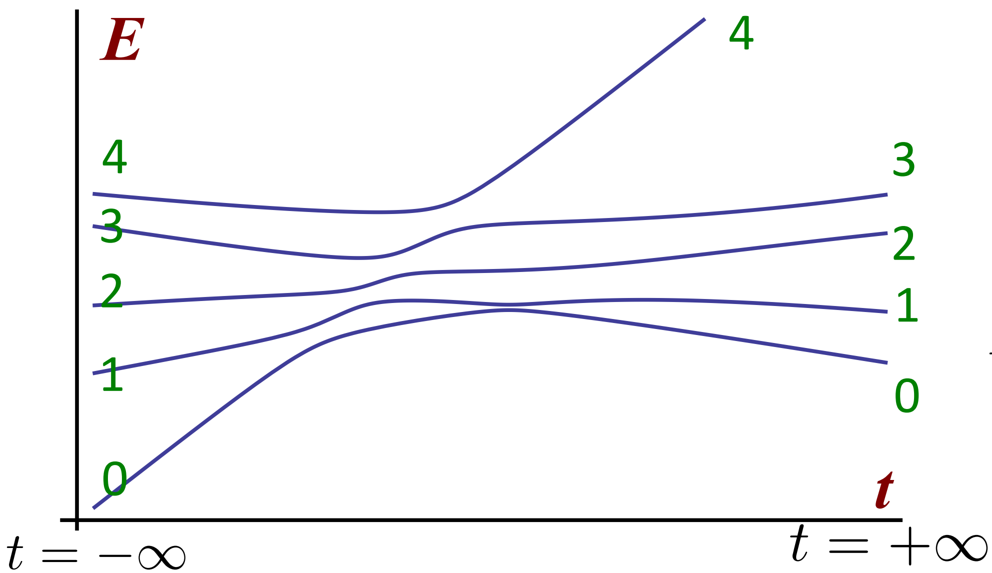
Poilblank et.al. Europhys. Lett. (1993)



Level spacing (*s*) distribution for Hubbard chain with 12 cites at ¼ filling, total momentum $P=\pi/6$, spin $S=0$

Time-dependent exactly solvable problems: multi-state Landau-Zener

$$H(t) = A + Bt, \quad A, B - N \times N \text{ Hermitian matrices}$$



$$p(0 \to k) = ?$$

***N*=2**: Landau-Zener formula

$$p(0 \to 1) = \exp\left(-\frac{2\pi |A_{12}|^2}{|b_1 - b_2|}\right)$$

Exact solution for N>2?

Only in very special cases

$$B = b\pi_{11}, \quad A = \sum_k a_i \pi_{kk} + \sum_{i \neq 1} v_k(\pi_{k1} + \pi_{1k}), \quad \pi_{ik} = |i\rangle\langle k|$$

Other nonequilibrium properties of integrable models: Mazur inequalities, no thermalization after a quantum quench – generalized Gibbs distribution

Notion of Quantum Integrability: What are we looking for?

Definition: Quantum Hamiltonian H_0 is integrable if…

Consequences:

1. Exact Solution (Yang-Baxter equation)
2. Commuting integrals $[H_i, H_j]=0; i, j=0,1…$
3. Energy level crossings
4. Poisson level statistics
5. Generalized Gibbs distribution for dynamics

Classical integrability

Definition: A classical Hamiltonian $H_0(p_k, q_k)$ with n degrees of freedom (n coordinates) is integrable if it has the maximum possible number (n) of independent Poisson-commuting integrals $\{H_i, H_j\}=0;\ i,j=0,1\ldots n$

⇩

Consequences:

1. Exact solution: the dynamics of $H_i(p_k, q_k)$ is exactly solvable by quadratures (Liouville-Arnold theorem)
2. Poisson level statistics semi-classically *[Berry & Tabor (1976)]*
3. Generalized Gibbs (dynamics uniform on invariant tori) *[E.Y., unpublished]*

Can we develop a similar sound notion of integrability in Quantum Mechanics – for N x N Hermitian matrices (Hamiltonians)?

Difficulties:

- ✓ Integrals of motion not well-defined, every Hamiltonian has a full set of commuting partners. What's an independent integral?

$$H_0 = \sum E_n |n\rangle\langle n|, \quad H_k = |k\rangle\langle k|, \quad [H_i, H_j] = 0$$

- ✓ No notion of # of degrees of freedom – how many integrals are needed?

Alternative definitions based on:

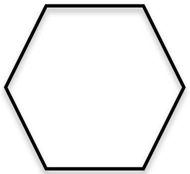
- ➢ Poisson level statistics or level xings – not exclusive to integrable models. Certain integrable systems don't have these e.g. Richardson (BCS) model
- ➢ Exact solution – but every matrix is "exactly solvable" in some sense

$$\det(H - \lambda I) = 0$$

- ➢ Plus, like in CM, would like these as consequences rather than definitions

Proposed solution: consider parameter-dependent Hamiltonians

Hints from Hubbard study, *u=U/T*:
Yuzbashyan, Altshuler, Shastry (2002)



Let $H(u) = T + uV$

u – real parameter,
T,V – *N* x *N* Hermitian matrices

Suppose we search for a commuting partner $H_1(u) = T_1 + uV_1$ also linear in u

$$[H_0(u), H_1(u)] = 0$$

$$\Updownarrow$$

$$[T, T_1] = [V, V_1] = 0, \quad [T, V_1] = [T_1, V]$$

Now these commutation relations severely constraint matrix elements of *T & V*. For a generic/typical *H(u)* – no commuting partners except the trivial one – a linear combination of itself and identity

N x N Hamiltonians linear in a parameter separate into two distinct classes

$H(u) = T + uV$

No commuting partners linear in u other than itself and identity (typical) – nonintegrable, $N^2/2$ real parameters are need to specify $H(u)$

Nontrivial commuting partners $H_k(u)=T_k+uV_k$ exist – integrable, turns out less than $4N$ parameters are needed – measure zero in the space of linear Hamiltonians

Classification by the number n of commuting partners

- $n = N$ **(maximum possible) – type 1 integrable system**
- $n = N\text{-}1$ **– type 2**
- $n = N\text{-}2$ **– type 3**
- …
- $n = N\text{-}M\text{+}1$ **– type M**
- …

Definition: A Hamiltonian operator $H \equiv H_0(u) = T_0 + uV_0$ is integrable if it has $n \geq 1$ nontrivial linearly independent commuting partners $H_i(u) = T_i + uV_i$

Owusu, Wagh, Yuzbashyan (2008)

$[H_i(u), H_j(u)] = 0$ for all u and $i, j = 0, \ldots, n-1$

General member of the commuting family: $h(u) = \sum_{i=1}^{n} d_i H_i(u)$

Examples of integrable models that fall under this definition:

- **1d Hubbard model: *u=U/T*, Hamiltonian and first integral are linear in *u***
- **integrable XXZ spin chain: *u* = anisotropy, $H_0(u)$ and $H_1(u)$ are linear in *u***
- **Gaudin magnets (all integrable pairing models): *u=B*=magnetic field, Hamiltonian and all integrals are linear in *u***

$$H_i(B) = B s_i^z + \sum_{k \neq i} \frac{\mathbf{s}_i \cdot \mathbf{s}_k}{\epsilon_i - \epsilon_k} \quad [H_i(B), H_j(B)] = 0$$

$\mathbf{s}_i$ – quantum spins ϵ_i – real parameters

What can we achieve with this notion of quantum integrability? – almost everything we wanted!!

1. Remarkably, we are able to explicitly construct integrable families with any given number *n* of integrals, i.e. resolve nonlinear commutation relations:

$$[H_i(u), H_j(u)] = 0 \Longleftrightarrow [T_i, T_j] = [V_i, V_j] = 0, \quad [T_i, V_j] = [T_j, V_i]$$

Owusu, Wagh, Yuzbashyan (2008)
Owusu, Yuzbashyan (2011)

Example: *n=N* (type 1 – max # of integrals)

$$H_i(u) = u\pi_{ii} + \sum_{k\neq i} \frac{\gamma_i\gamma_k(\pi_{ik}+\pi_{ki}) - \gamma_k^2\pi_{ii} - \gamma_i^2\pi_{kk}}{\epsilon_i - \epsilon_k}$$

$\pi_{ik} = |i\rangle\langle k|$ - projectors, γ_i, ϵ_i - arbitrary real numbers

Type 1 maps onto a sector of Gaudin magnets with rational spins

$$u = B, \quad \gamma_i = s_i^2, \quad |i\rangle = s_i^+|0\rangle$$

What can we achieve with this notion of quantum integrability? – almost everything we wanted!!

2. Exact solution through a single algebraic equation for all types (cf. Bethe Ansatz)

(type 1)

$$\sum_j \frac{\gamma_j^2}{\lambda - \epsilon_j} = u, \quad E_k = \frac{\gamma_k^2}{\lambda - \epsilon_k}, \quad |\lambda\rangle = \sum_j \frac{\gamma_j |j\rangle}{\lambda - \epsilon_j}$$

γ_j, ϵ_j - given; solve for λ

Owusu, Wagh, Yuzbashyan (2008)

3. Yang-Baxter formulation

scattering matrix

$$S_{ij} = \frac{(\epsilon_j - \epsilon_i)I + 2g\Pi_{ij}}{(\epsilon_j - \epsilon_i) + g\,(\gamma_i^2 + \gamma_j^2)}$$

$$S_{ik}S_{jk}S_{ij} = S_{ij}S_{jk}S_{ik}$$

Yuzbashyan, Shastry (2011)

4. Can prove the existence of level xings and determine their number as a function of the # (*n*) of commuting partners in an integrable family

$$\max \# \text{ of crossings} = (N^2 - 5N + 2)/2 + n$$

Owusu, Yuzbashyan (2011)

5. Poisson level statistics except at isolated points of measure zero in the parameter space

Hansen, Yuzbashyan, Shastry (in progress)

Applications: Blocks of 1d Hubbard model (6 sites, 3 up and 3 down spins)

- Each block is characterized by a complete set of quantum #s (***P***, S^2, S_z…)
- We determine the type of each block

# of nontrivial integrals = Size – Type – 1

Momenta $P = \pi/6, 5\pi/6$	
Size of the block	Its Type
8×8	Type 3
3×3	Type 1
16×16	Type 12
14×14	Type 3
3×3	Type 1

Momenta $P = \pi/3, 2\pi/3$	
Size of the block	Its Type
12×12	Type 7
14×14	Type 11
4×4	Type 1
2×2	—
16×16	Type 6

Results for Hubbard:

- In most blocks – exact solution in terms of a single equation – vast simplification over Bethe Ansatz (9 equations)!

- New symmetries in 1d Hubbard! # of nontrivial integrals linear in u=U/T is 14-3-1=10. Only one such integral was identified before

Solvable multi-state Landau-Zener problem turns out to be a special case of Type1!

N **state Landau -Zener problem:** $p(0 \to k) = ?$

$$H(t) = A + Bt, \quad A, B - N \times N \text{ Hermitian matrices}$$

Exact solution known only in very special cases

$$B = b\pi_{11}, \quad A = \sum_k a_i \pi_{kk} + \sum_{i \neq 1} v_k(\pi_{k1} + \pi_{1k}), \quad \pi_{ik} = |i\rangle\langle k|$$

But this is just one of Type 1 basic operators

$$H_i(u) = u\pi_{ii} + \sum_{k \neq i} \frac{\gamma_i \gamma_k (\pi_{ik} + \pi_{ki}) - \gamma_k^2 \pi_{ii} - \gamma_i^2 \pi_{kk}}{\epsilon_i - \epsilon_k}$$

Set $u = bt, \; i = 1, \; \epsilon_1 = 0, \; \gamma_1 = 1, \; \epsilon_k = -1/a_k, \; \gamma_k = v_k/a_k$

General Type 1: $h(u) = \sum_i d_i H_i(u)$

Is it possible to solve multi-state Landau-Zener for a much larger class of Hamiltonians – general Type 1 and other Types???

Summary:

- ✓ **Proposed a simple, natural notion of quantum integrability based on parameter-dependence**
- ✓ **Derived exact solution, existence of level crossings (and their #), Yang-Baxter formulation from this notion**
- ✓ **Exact solution is in terms of a single algebraic equation implying that at least in some cases Bethe's Ansatz equations can be dramatically simplified**
- ✓ **New linear integrals in the 1d Hubbard model**
- ✓ **Exact solution of the multi-state Landau-Zener problem for a new, much wider class of Hamiltonians?**