Order and disorder in SU(3) and SU(4) Heisenberg systems

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Correlations and coherence in quantum systems

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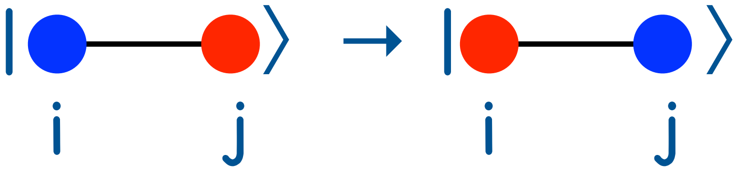
What are the SU(N) symmetric Heisenberg models that we are interested in?

$$\mathcal{H} = \sum_{i,j} \mathcal{P}_{i,j}$$

$\mathcal{P}_{i,j}$ is the transposition operator

$\mathcal{P}_{i,j}$



N species on each site that are treated equally.

$$\mathcal{P}_{ij}|\beta_i\alpha_j\rangle = |\alpha_i\beta_j\rangle$$

simplest example:
SU(2) S=1/2 (fundamental representation)
[but not the S=1 !]

Why do we care about SU(3) or SU(4) Heisenberg models?

(i) They are interesting and intelectually challenging (funding agencies ?)

(ii) Nostalgia from our younghood, when we all wanted to become particle physicist (SU(3) and quarks)

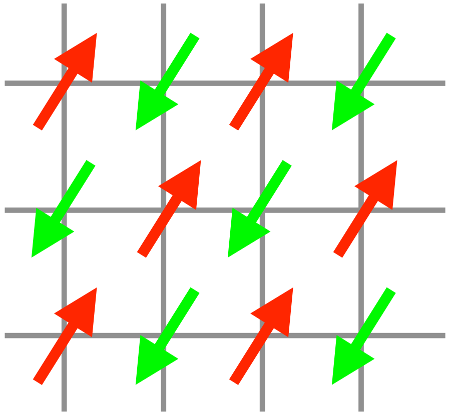
(iii) Cold atomic gases

(iv) Spin models

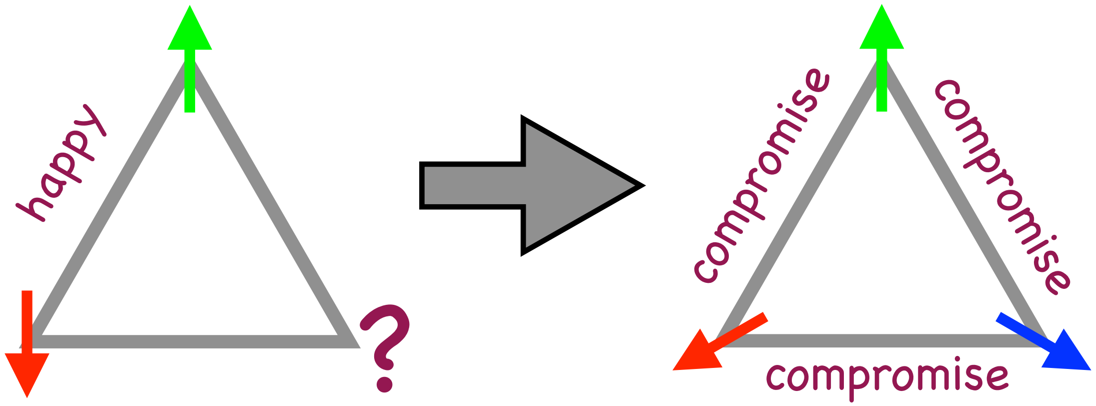
(v) Spin-orbital models (SU(4))

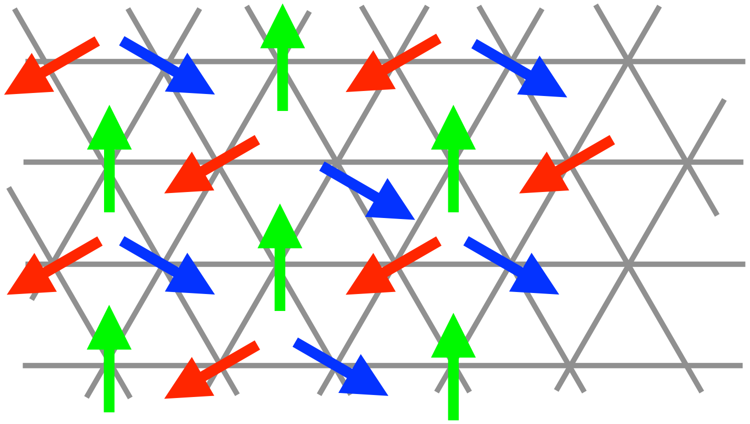
SU(2) quantum Heisenberg models

square and honeycomb
(bipartite) lattices:
happy (mean field)
antiferromagnets



triangular lattice:
frustrated antiferromagnet



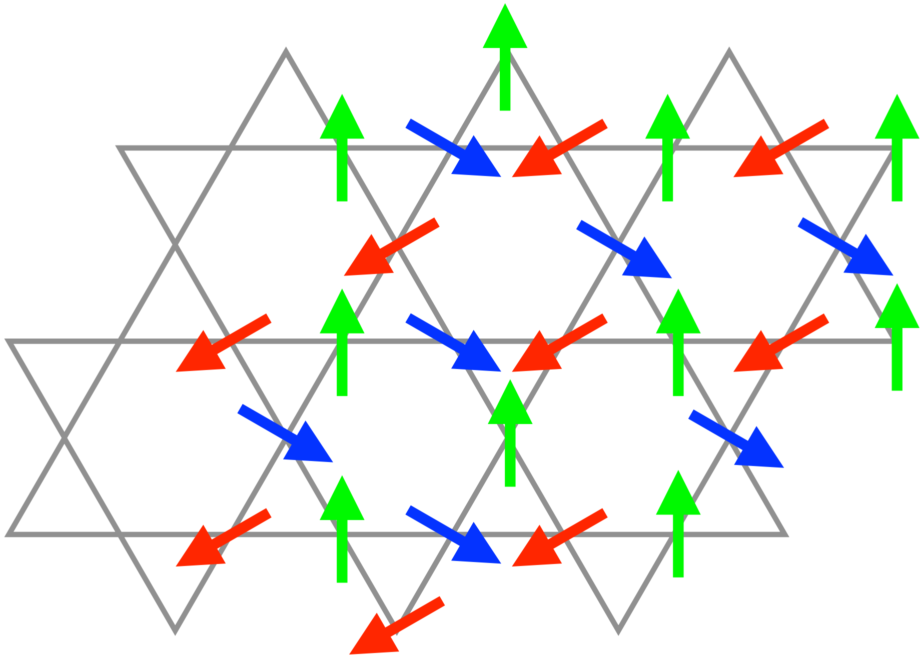


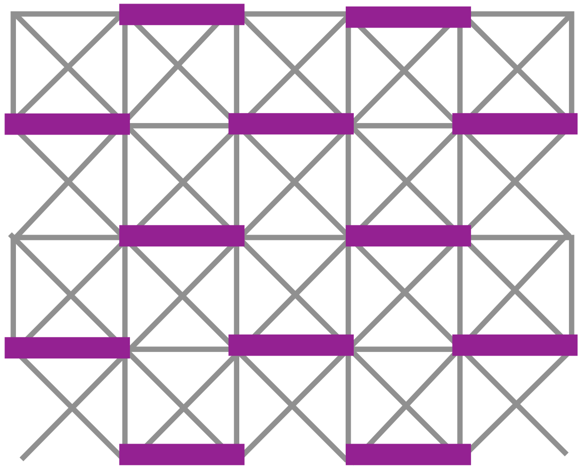
3-sublattice 120 degree state

SU(2) quantum Heisenberg models

kagome lattice: highly frustrated antiferromagnet, mean field ground state macroscopically degenerate.

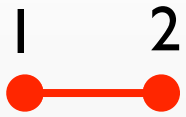
Ground state debated, likely a Z2 spin liquid





frustrated square lattice: dimerized state

SU(2) vs. SU(3) - two sites



$\mathcal{P}_{12}(|\alpha\beta\rangle - |\beta\alpha\rangle) = -(|\alpha\beta\rangle - |\beta\alpha\rangle)$ E=−1 , odd wave function

$\mathcal{H} = \mathcal{P}_{12}$ $\mathcal{P}_{12}(|\alpha\beta\rangle + |\beta\alpha\rangle) = +(|\alpha\beta\rangle + |\beta\alpha\rangle)$ E=+1, even wave function

Addition of two S=½ SU(2) spins:

½ ⊗ ½ = 0 ⊕ 1

using Young diagrams:

2 × 2 = 1 + 3

$\square \otimes \square = \begin{array}{c}\square\\ \square\end{array} \oplus \square\square$

$\square$ ↑ or ↓ spin

$\begin{array}{c}\square\\ \square\end{array}$ |↑↓⟩−|↓↑⟩ singlet, odd (anti-symmetrical)

$\square\square$ |↑↑⟩, |↑↓⟩+|↓↑⟩, |↓↓⟩ triplet even (symmetrical)

Addition of two SU(3) spins:

3 × 3 = 3 + 6

$\square \otimes \square = \begin{array}{c}\square\\ \square\end{array} \oplus \square\square$

$\square$ |a⟩, |b⟩, and |c⟩.

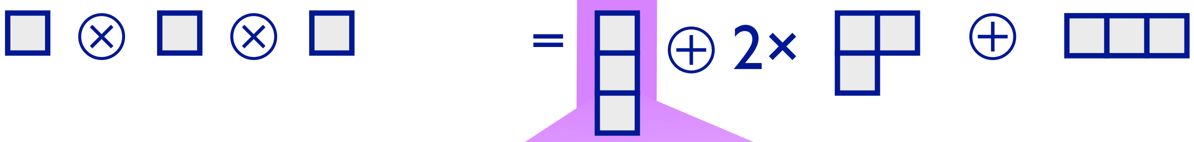
$\begin{array}{c}\square\\ \square\end{array}$ |ab⟩−|ba⟩, |ac⟩−|ca⟩, |bc⟩−|cb⟩: odd (anti-symmetrical).

$\square\square$ |aa⟩, |bb⟩, |cc⟩, |ab⟩+|ba⟩, |ac⟩+|ca⟩, and |bc⟩+|cb⟩. even (symmetrical)

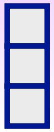
SU(3) irreps on 3 sites

Addition of three SU(3) spins (27 states):

$$3 \times 3 \times 3 = 1 + 2 \times 8 + 10$$

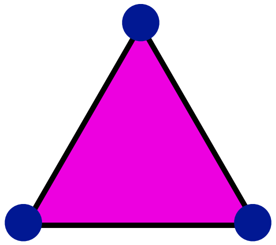


SU(3) singlet



$$= |ABC\rangle + |CAB\rangle + |BCA\rangle - |BAC\rangle - |ACB\rangle - |BCA\rangle$$

spins fully antisymmetrized



in the SU(3) singlet the spins are fully entangled:
we cannot write it in a product form

What methods do we use?

(i) Variational - site factorized wave function
(ii) Flavor wave calculations
(iii) Exact diagonalization of small clusters
(iv) iPEPS: infinite project entangled pair states(variational approach based on tensor ansatz)
(v) Variational: Gutzwiller projected fermionic wave functions

Variational (classical) approach

a site-product wave function for SU(3):

$$|\Psi\rangle = \prod_i |\psi_i\rangle$$

$$|\psi_i\rangle = d_{A,i}|A\rangle_i + d_{B,i}|B\rangle_i + d_{C,i}|C\rangle_i$$

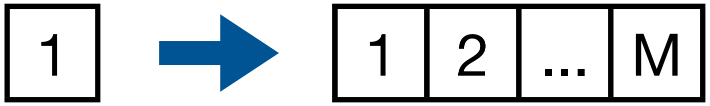
$$E_{\mathrm{var}} = \frac{\langle\Psi|\mathcal{H}|\Psi\rangle}{\langle\Psi|\Psi\rangle} = J\sum_{\langle i,j\rangle}\left|\mathbf{d}_i\cdot\bar{\mathbf{d}}_j\right|^2$$

minimal, when the $\mathbf{d}_i$ and $\mathbf{d}_j$ on the bond are orthogonal

two different colors on a bond

SU(3) flavour-wave theory

N. Papanicolaou, Nucl. Phys. B **305**, 367 (1988)
A. Joshi *et al.* PRB **60**, 6584 (1999)



We enlarge the fundamental to the fully symmetric representation of M boxes.

States in the fully symmetric multiplet can be represented by 3 Schwinger bosons a_A,a_B and a_C

$$a^\dagger_{A,i}a_{A,i} + a^\dagger_{B,i}a_{B,i} + a^\dagger_{C,i}a_{C,i} = M$$

$$\mathcal{P}_{ij} = \sum_{\mu,\nu\in\{A,B,C\}} a^\dagger_{\mu,i}a^\dagger_{\nu,j}a_{\nu,i}a_{\mu,j}$$

The site product wave function is the “classical” solution (no quantum entanglement between sites)

$$|\Psi\rangle = \prod_i |\psi_i\rangle$$

$$|\psi_i\rangle = d_{A,i}|A\rangle_i + d_{B,i}|B\rangle_i + d_{C,i}|C\rangle_i$$

cf. SU(2) spin coherent state for SU(2)

$$|\vartheta,\varphi\rangle = \cos\frac{\vartheta}{2}|\uparrow\rangle + \sin\frac{\vartheta}{2}e^{i\varphi}|\downarrow\rangle$$

SU(3) flavour-wave theory

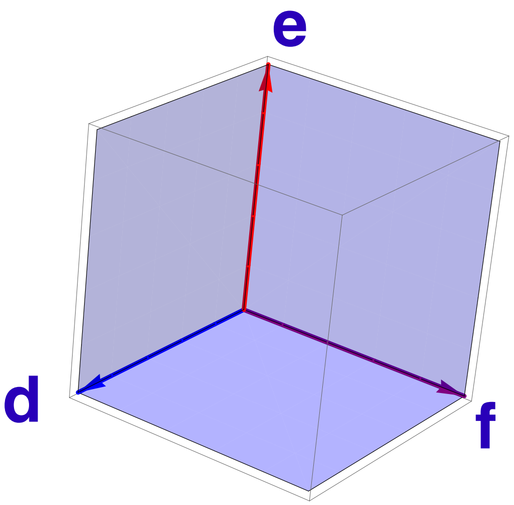
Local coordinate system

$$|\Psi\rangle = \prod_i \left(\tilde{a}_i^\dagger\right)^M |\tilde{0}\rangle$$

$$\tilde{a}^\dagger_{A,j} = \sum_\mu d_{\mu,j} a^\dagger_{\mu,j}$$

$$\tilde{a}^\dagger_{B,j} = \sum_\mu e_{\mu,j} a^\dagger_{\mu,j}$$

$$\tilde{a}^\dagger_{C,j} = \sum_\mu f_{\mu,j} a^\dagger_{\mu,j}$$



1/M expansion:

$$\tilde{a}^\dagger_{A,i}, \tilde{a}_{A,i} \to \sqrt{M - \tilde{a}^\dagger_{B,i}\tilde{a}_{B,i} - \tilde{a}^\dagger_{C,i}\tilde{a}_{C,i}}$$

$$\to \sqrt{M} - \frac{1}{2\sqrt{M}}\left(\tilde{a}^\dagger_{B,i}\tilde{a}_{B,i} + \tilde{a}^\dagger_{C,i}\tilde{a}_{C,i}\right) + \ldots$$

Holstein-Primakoff bosons

$$\mathcal{P}_{ij} \quad \to \quad M\left[(\bar{\mathbf{e}}_i \cdot \mathbf{d}_j)\tilde{a}^\dagger_{B,i} + (\bar{\mathbf{f}}_i \cdot \mathbf{d}_j)\tilde{a}^\dagger_{C,i} + (\bar{\mathbf{d}}_i \cdot \mathbf{e}_j)\tilde{a}_{B,j} + (\bar{\mathbf{d}}_i \cdot \mathbf{f}_j)\tilde{a}_{C,j}\right]$$
$$\times\left[(\mathbf{e}_i \cdot \bar{\mathbf{d}}_j)\tilde{a}_{B,i} + (\mathbf{f}_i \cdot \bar{\mathbf{d}}_j)\tilde{a}_{C,i} + (\mathbf{d}_i \cdot \bar{\mathbf{e}}_j)\tilde{a}^\dagger_{B,j} + (\mathbf{d}_i \cdot \bar{\mathbf{f}}_j)\tilde{a}^\dagger_{C,j}\right] - M$$

quadratic in operators: we know how to diagonalize it (spin wave)

$$\mathcal{H} = -MJL + M\sum_\nu \sum_{\mathbf{k}} \omega_\nu(\mathbf{k})\left(\alpha^\dagger_\nu(\mathbf{k})\alpha_\nu(\mathbf{k}) + \frac{1}{2}\right)$$

The models

(i) SU(3) model on the triangular lattice

(ii) SU(3) model on the square lattice

(iii) SU(4) model on the square and cubic lattice

(iv) SU(4) model on the honeycomb lattice

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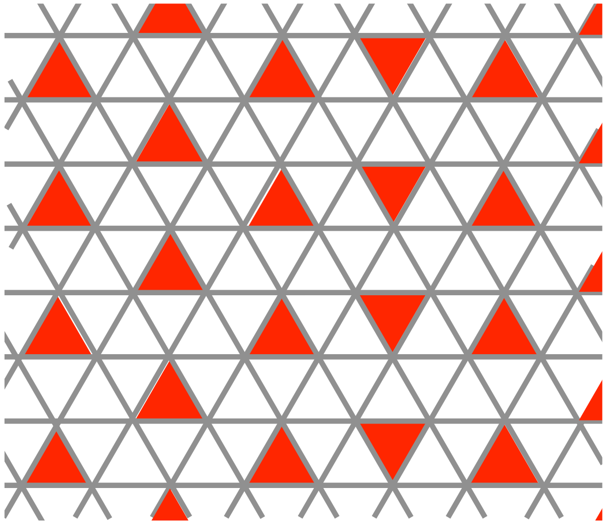
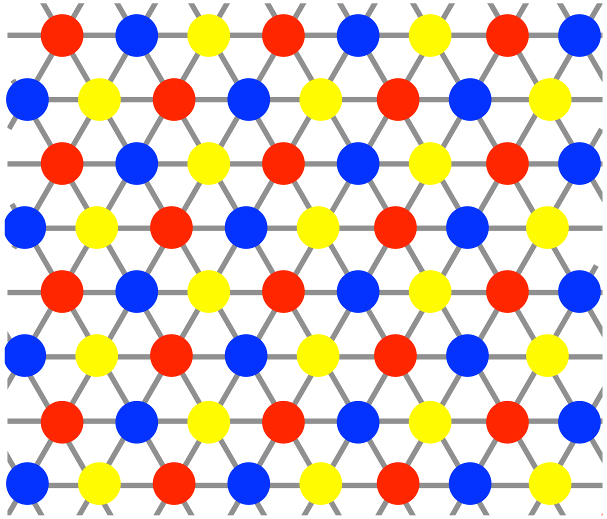
The fate of SU(3) on triangular lattice



crystal of singlets?

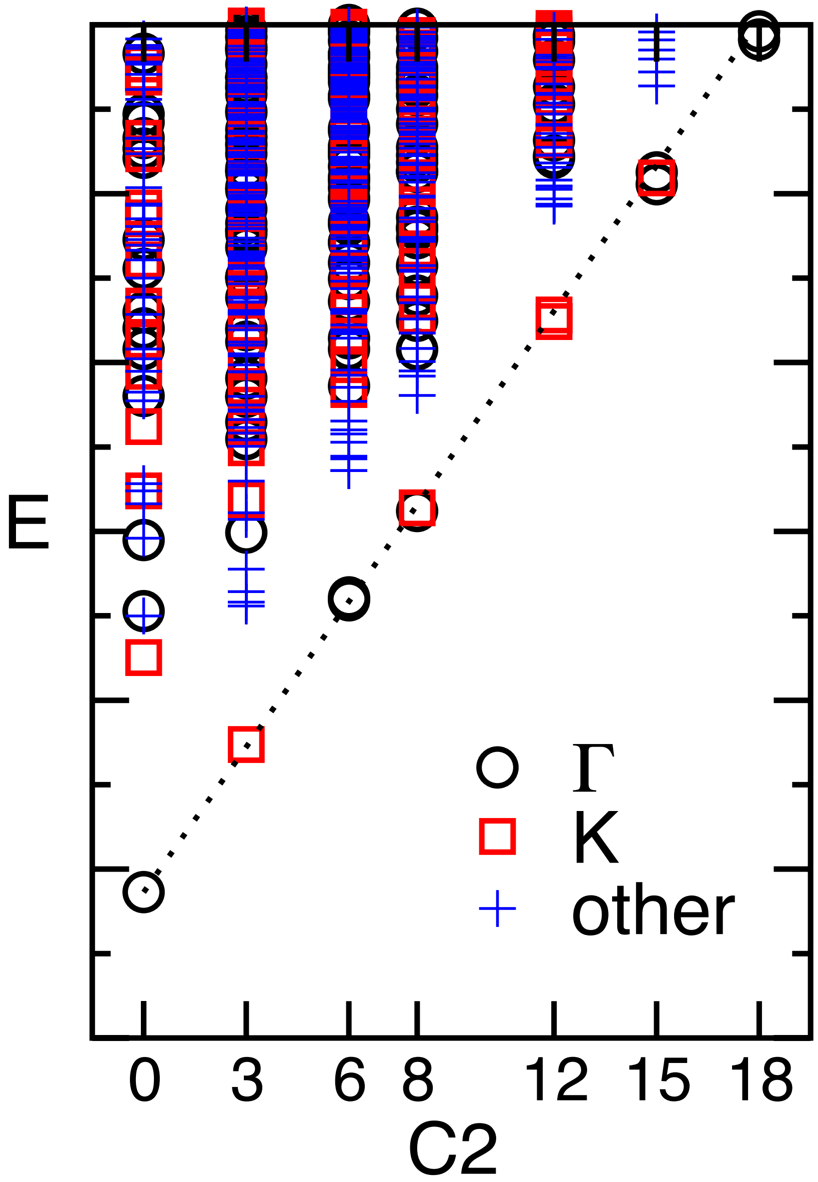


“classical” solution?

SU(3) classical state is perfectly happy on the triangular lattice - the 3 mutually perpendicular **d**’s form a 3 sublattice structure.

SU(2) frustrated!

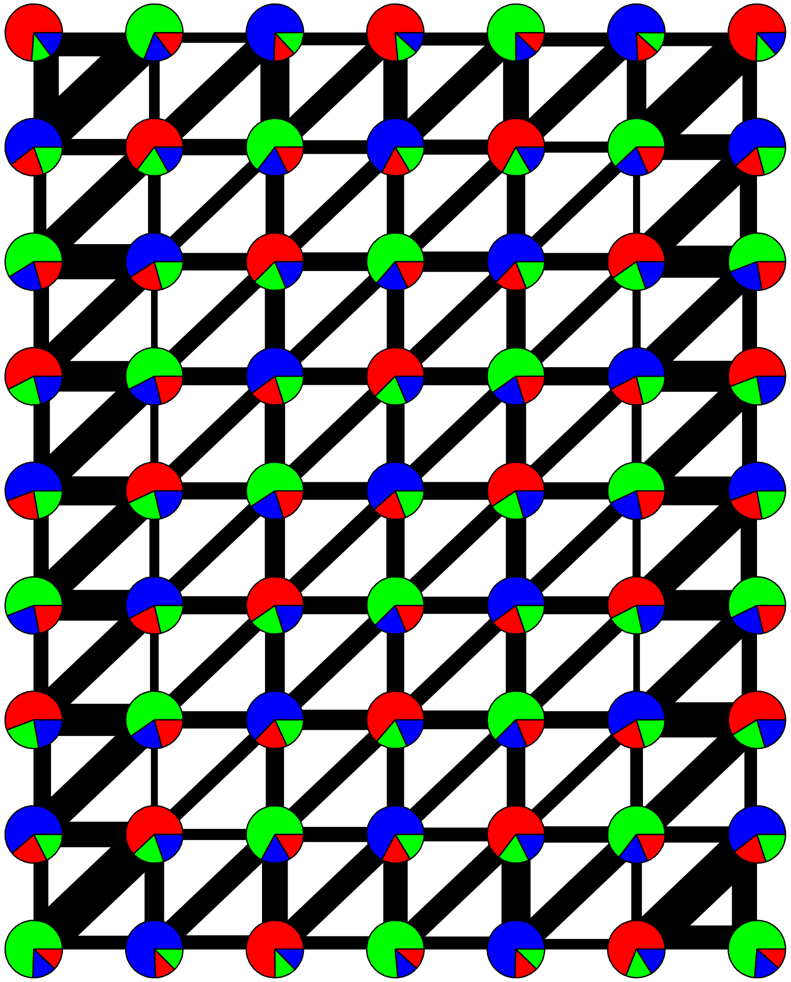
SU(3) on triangular lattice - exact diagonalization



Signature of SU(3) breaking in the excitation spectrum: Anderson towers compatible with 3 sublattice order

C2 - Casimir operator, analog of the total spin S^2

Unbiased calculation: iPEPS, DMRG



B. Bauer, P. Corboz, A. M. Läuchli, L. Messio,
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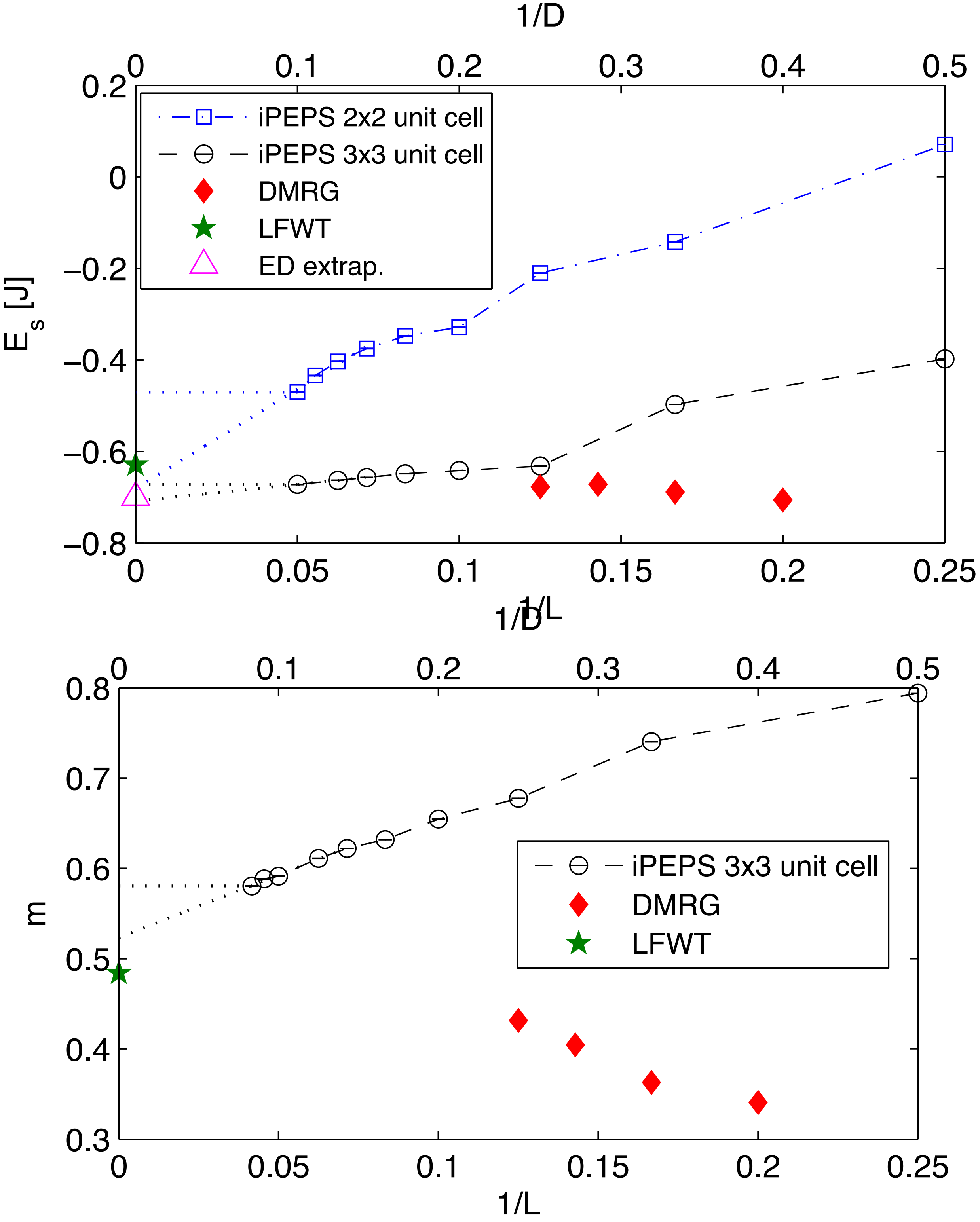
Phys. Rev. B **85**, 125116/1-11 (2012)



The models

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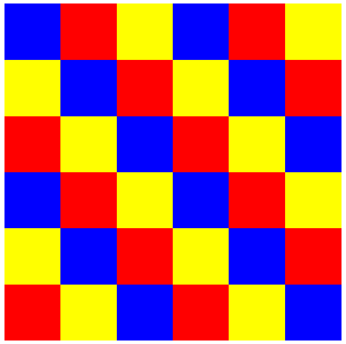
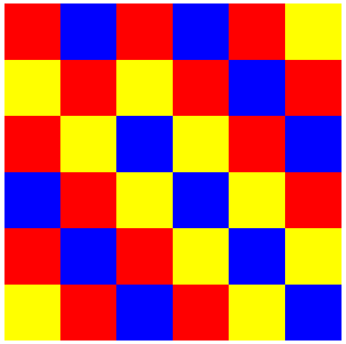
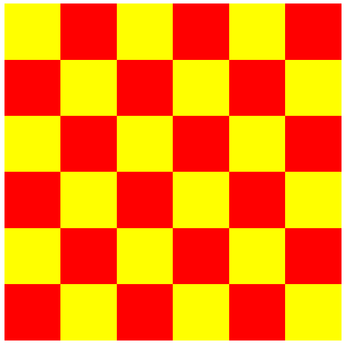
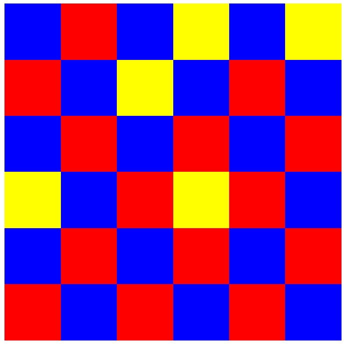
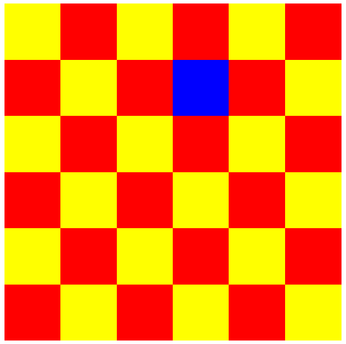
SU(3) classical solutions: macroscopically degenerate



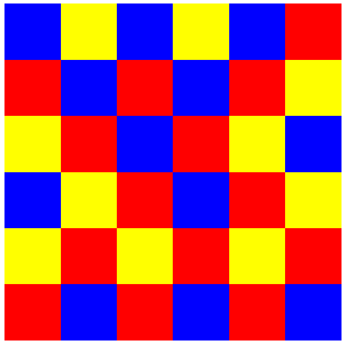








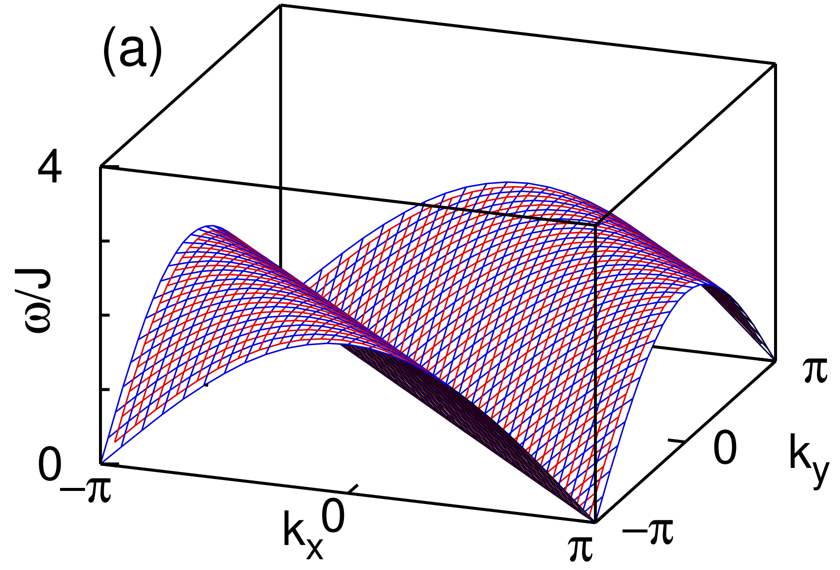
All bonds happy at the mean field level,
frustration due to abundance of choices

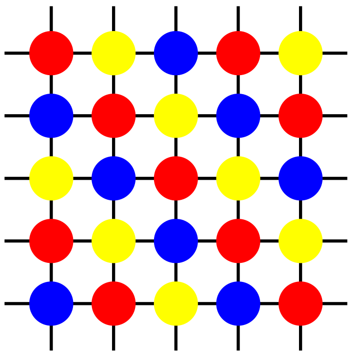


Order by disorder:
quantum fluctuations select the
ground state

$$E_{ZP} = \frac{M}{2}\sum_{\nu}\sum_{\mathbf{k}}\omega_{\nu}(\mathbf{k})$$

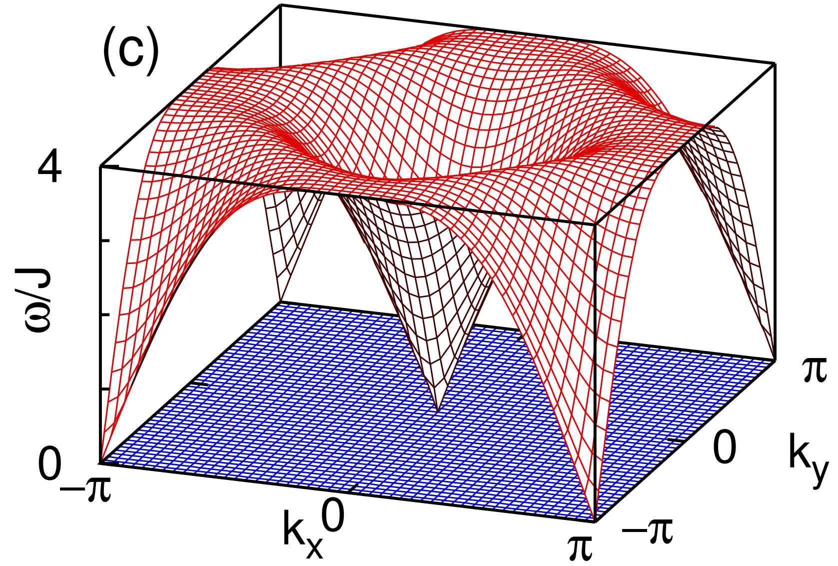
SU(3) flavour-wave: dispersions, zero point energy

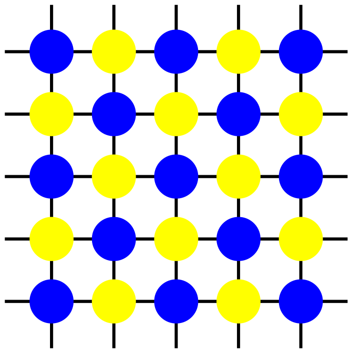




$E_{ZP} = 1.27$

The winner !



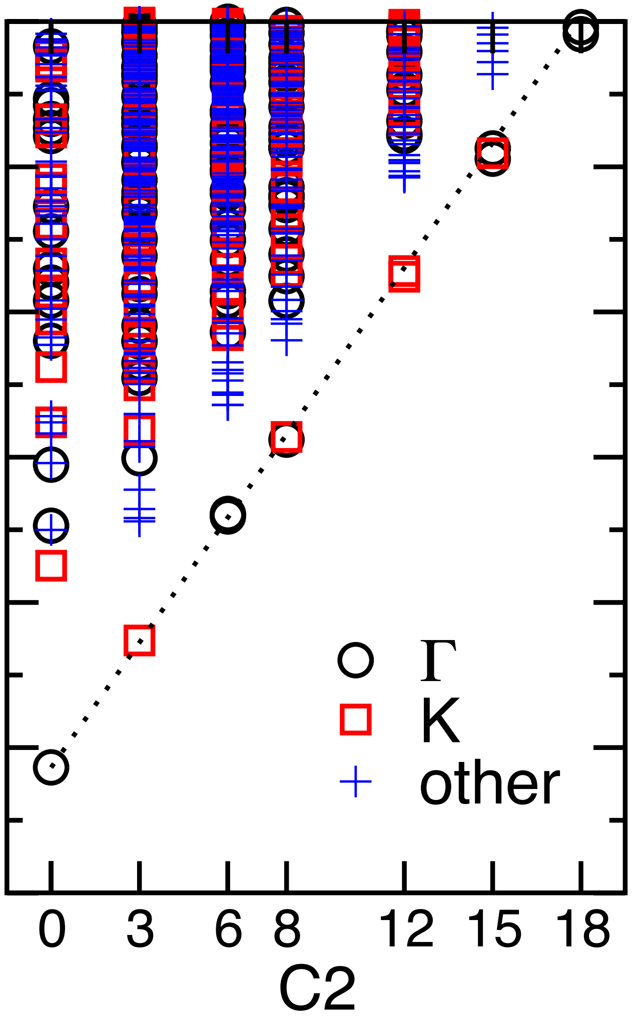


$E_{ZP} = 1.68$

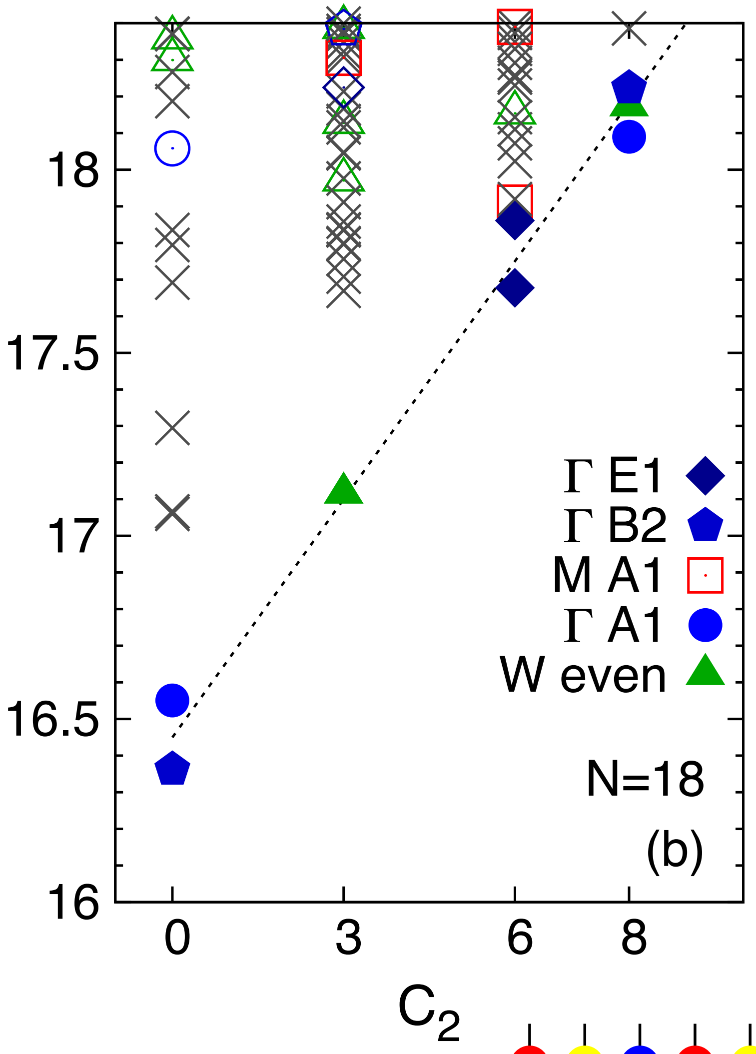
T. A. Tóth, A. M. Läuchli, F. Mila, and K. Penc: PRL 105, 265301 (2010).

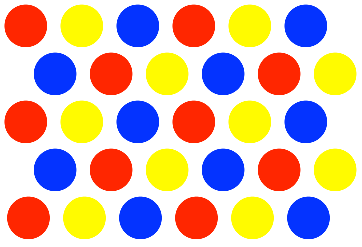
Unbiased calculation: ED, Anderson towers

triangular lattice

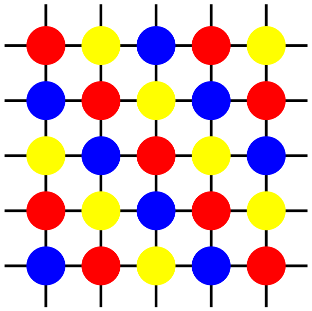


square lattice

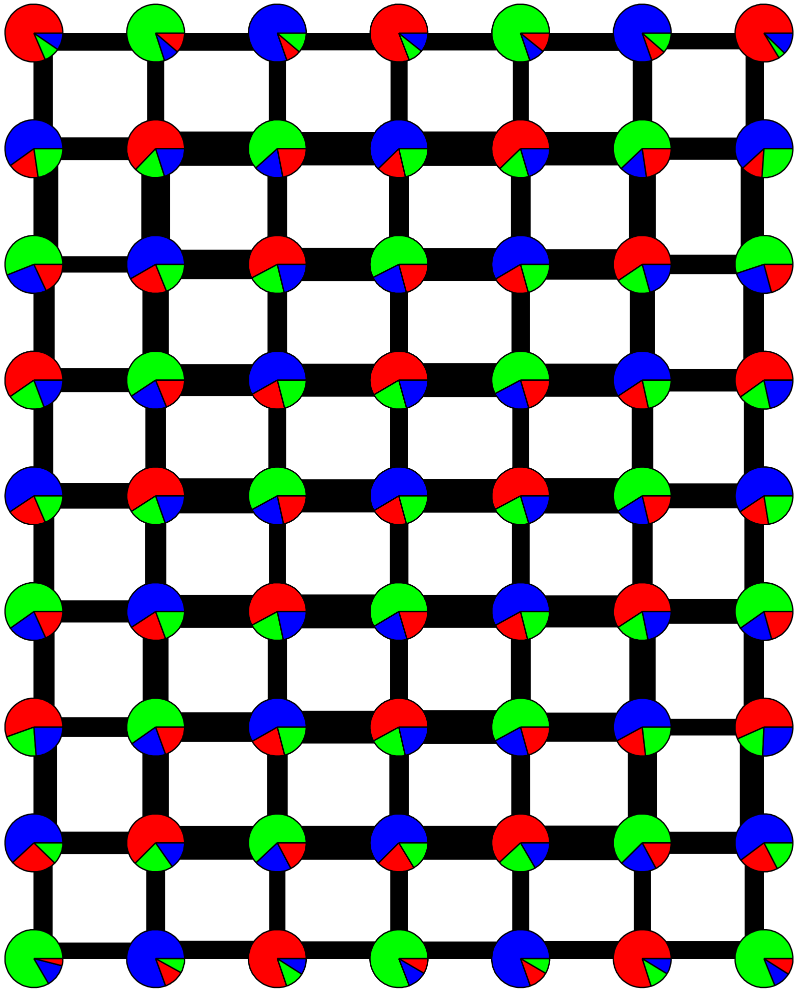


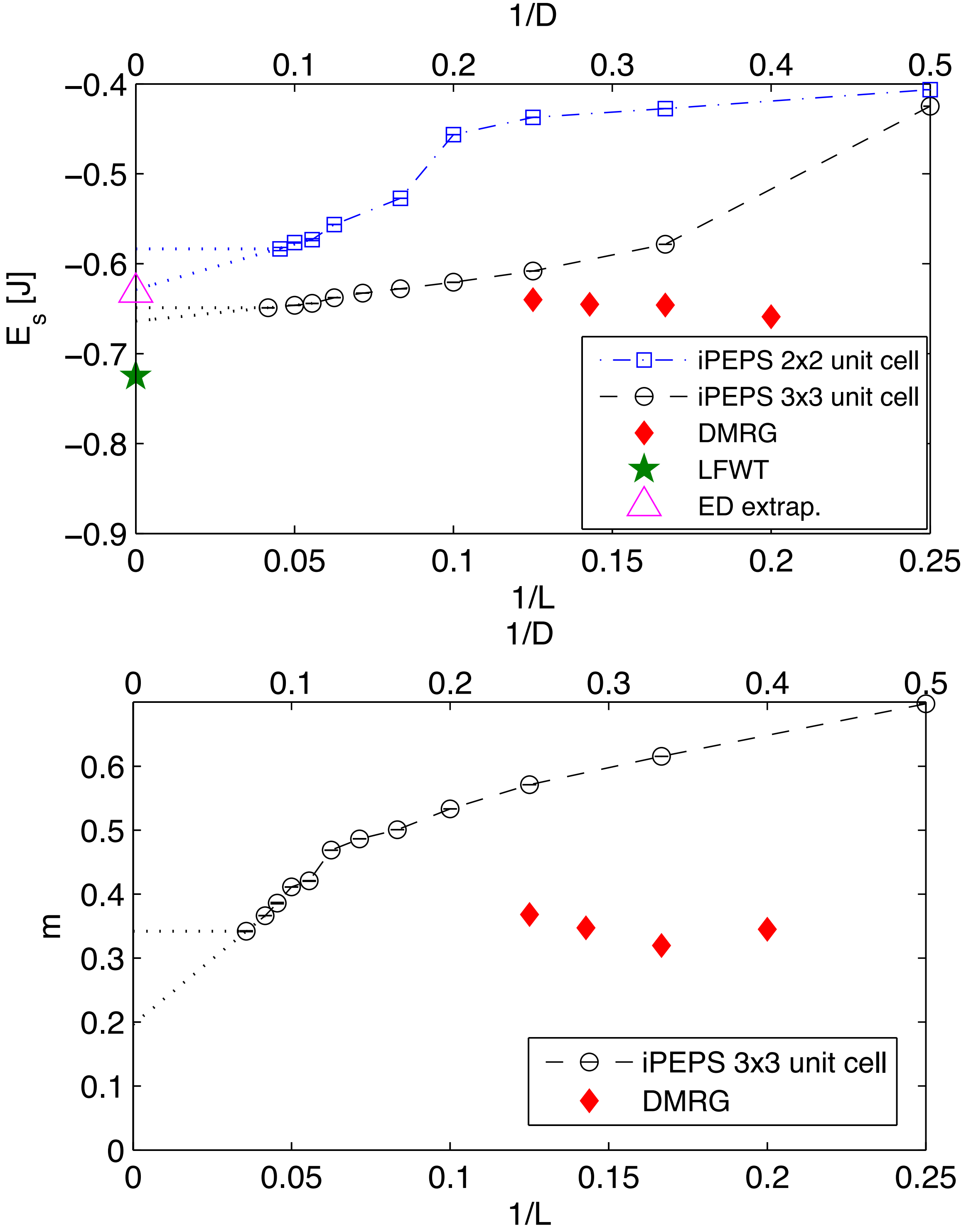


additional Z2 symmetry



Unbiased calculation: iPEPS, DMRG





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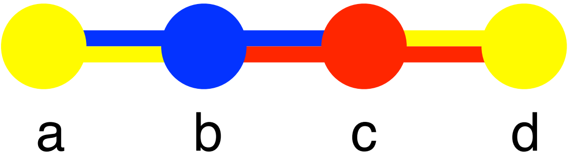
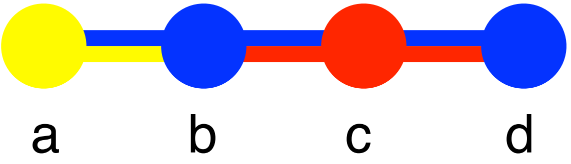
structure of the flavor wave Hamiltonian





$$\mathcal{H} = (a^\dagger + b)(b^\dagger + a) + (b^\dagger + c)(c^\dagger + b) + (c^\dagger + d)(d^\dagger + c)$$

each term separately $E_{ZP} = 0$

nearest neighbor also of different color

$$\mathcal{H} = (a^\dagger + b)(b^\dagger + a) + (b^\dagger + c)(c^\dagger + b) + (c^\dagger + d)(d^\dagger + c)$$

the 3-site term gives $E_{ZP} > 0$

The models

(i) SU(3) model on the triangular lattice

(ii) SU(3) model on the square lattice

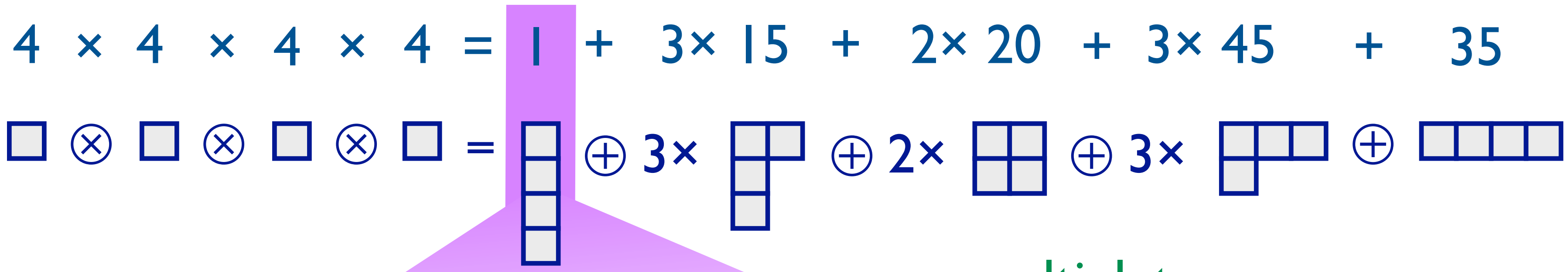
(iii) SU(4) model on the square and cubic lattice

(iv) SU(4) model on the honeycomb lattice

SU(4) irreps on 4 sites

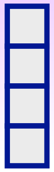
Addition of four SU(4) spins (256 states):

$$4 \times 4 \times 4 \times 4 = 1 + 3\times 15 + 2\times 20 + 3\times 45 + 35$$



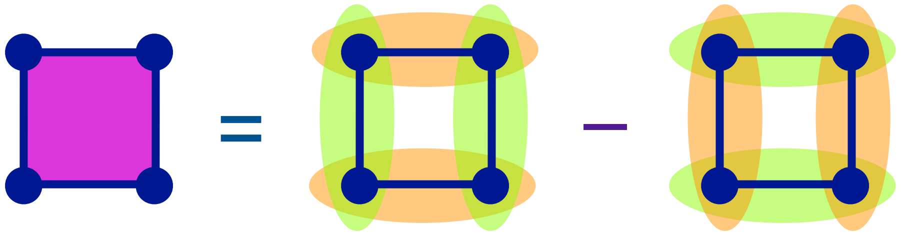
multiplets

SU(4) singlet



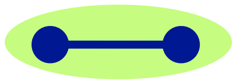
$= |abcd\rangle - |bacd\rangle + |badc\rangle - |bdac\rangle - \ldots$

spins fully antisymmetrized



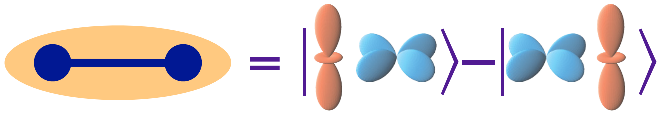
SU(4) singlet plaquette
entangled spins and orbitals

spin singlet bond:



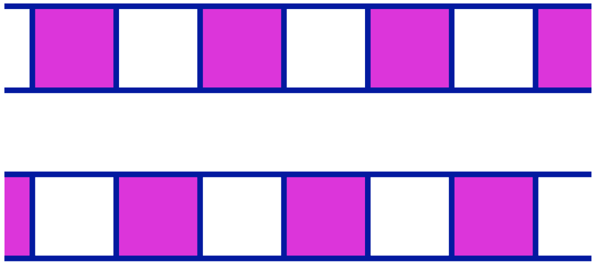
$= |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

orbital singlet bond:

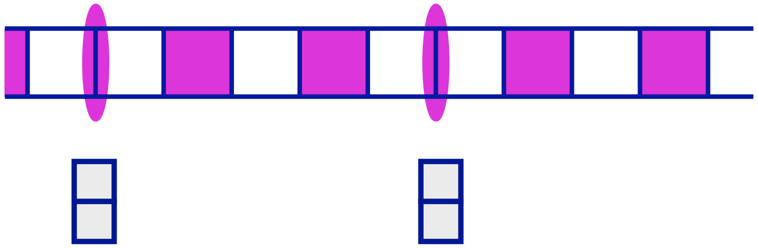


SU(4) ladder

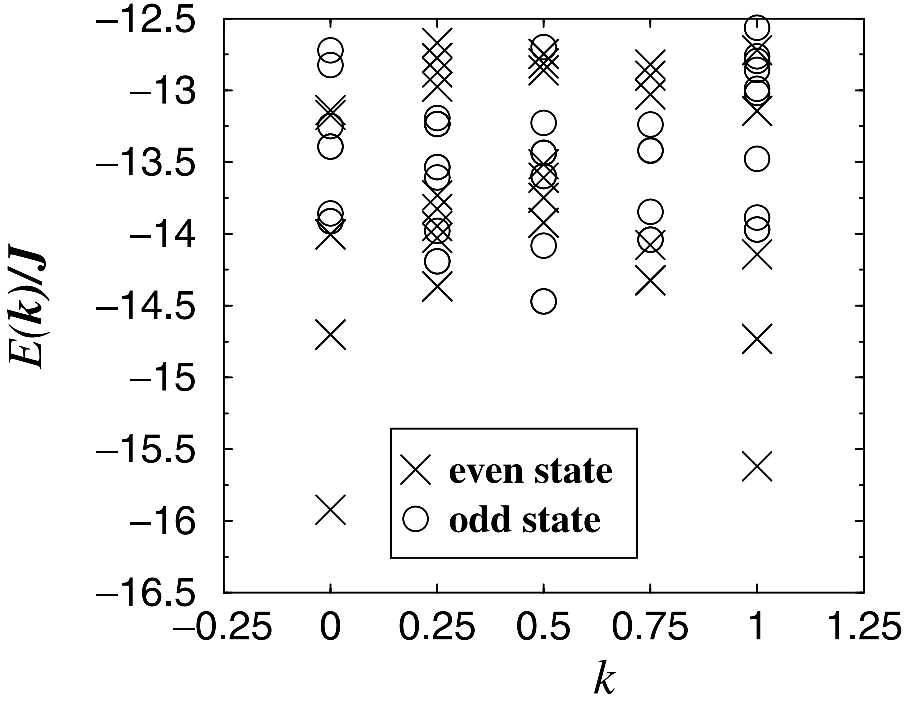
GS twofold degenerate
(translational invariance broken):



gapped excitation spectrum,
situation similar to
J_1-J_2 SU(2) Heisenberg chain:

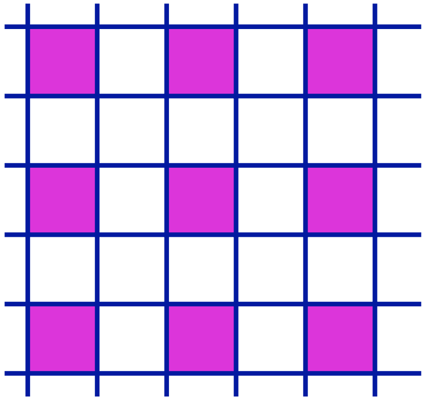


M. van den Bossche et al.,
Phys. Rev. Lett. **86**, 4124 (2001).



SU(4) on 2D-square lattice

Ground state 4-fold degenerate?

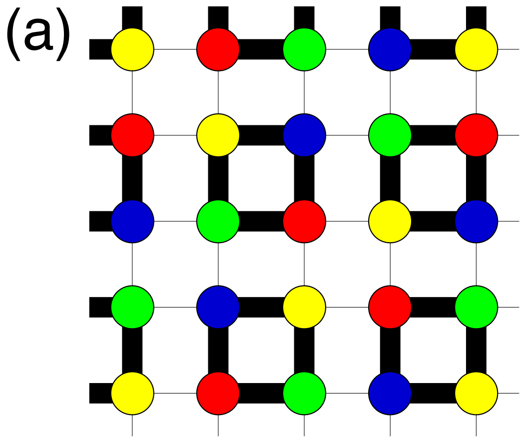


Z2 liquid ?
Wang & Viswanath (PRB 2009)

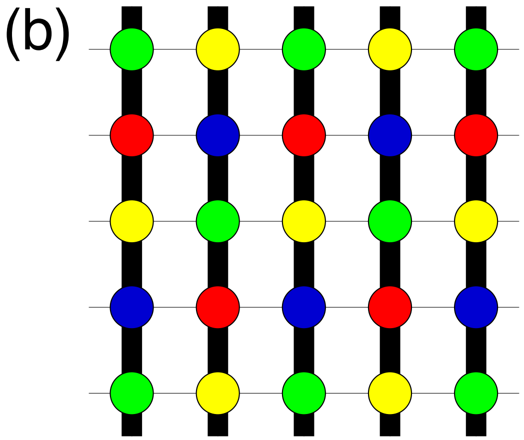
SU(4) on 2D-square lattice: breaking SU(4)?

The number of good mean field solution is highly degenerate.

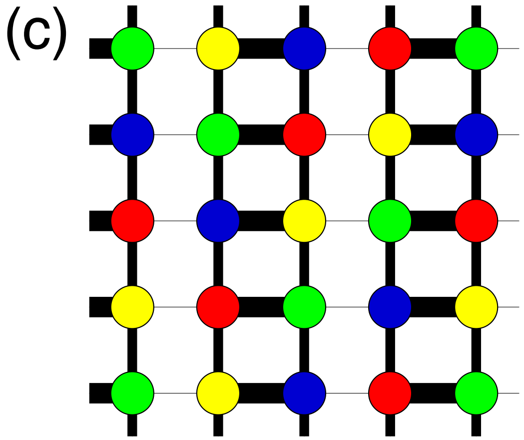
Order by disorder: Assuming on-site long range order, which configuration has the lowest zero-point energy?



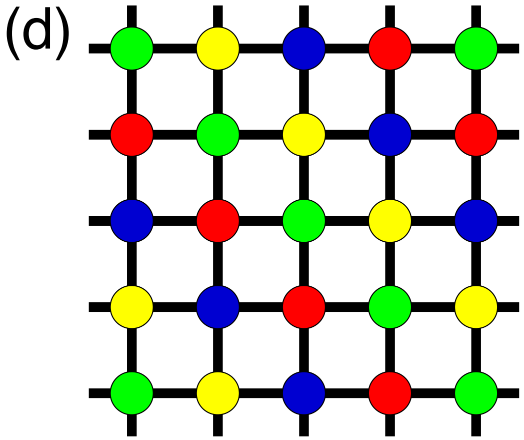
E/N = –3/2 J



E/N = –1.363 J

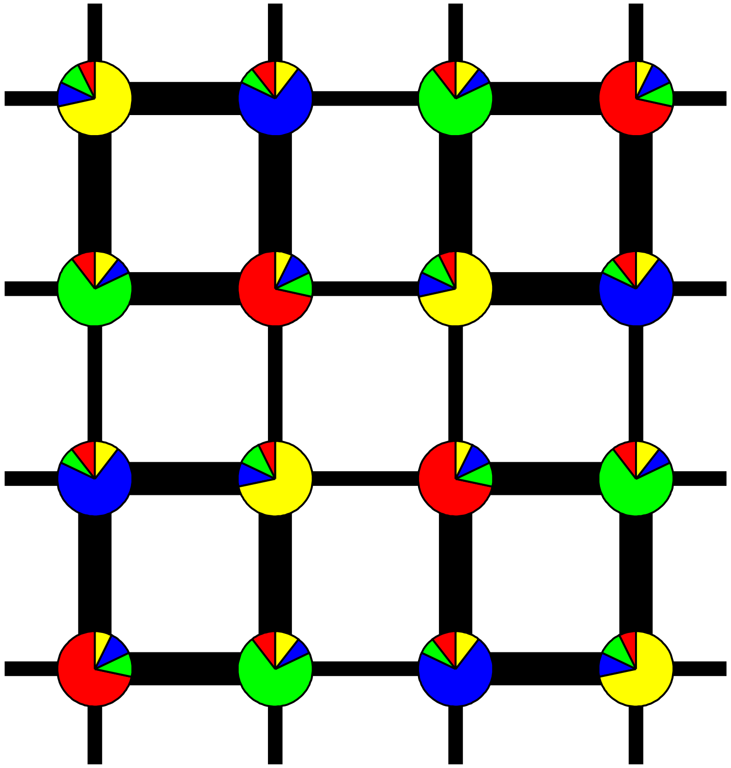


E/N = –1.293 J



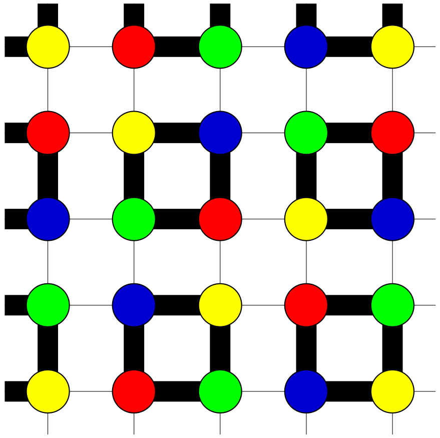
E/N = –0.727 J

SU(4) on 2D-square lattice: iPEPS

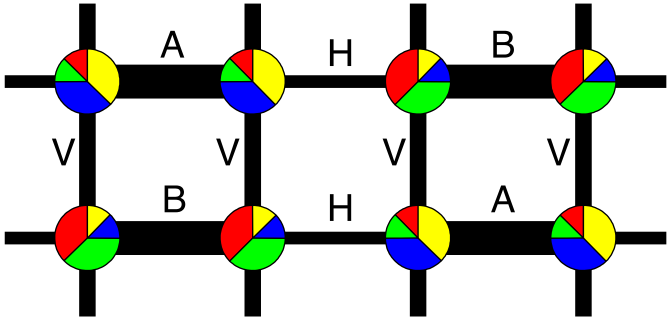


Tensor network method, with D-dimensional links between sites.

D = 2 and a unit cell 4 × 4. The bond- energy pattern of the ground state is similar to that of the plaquette state selected by LFWT.



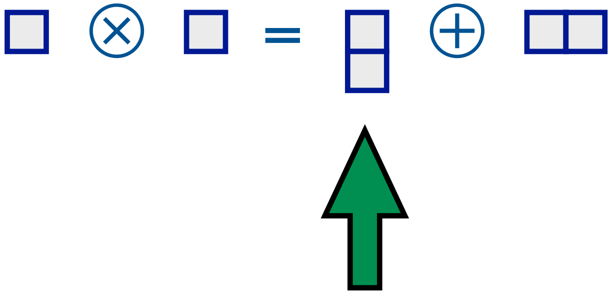
SU(4) on 2D-square lattice: iPEPS



D = 12 and a unit cell 4 × 2

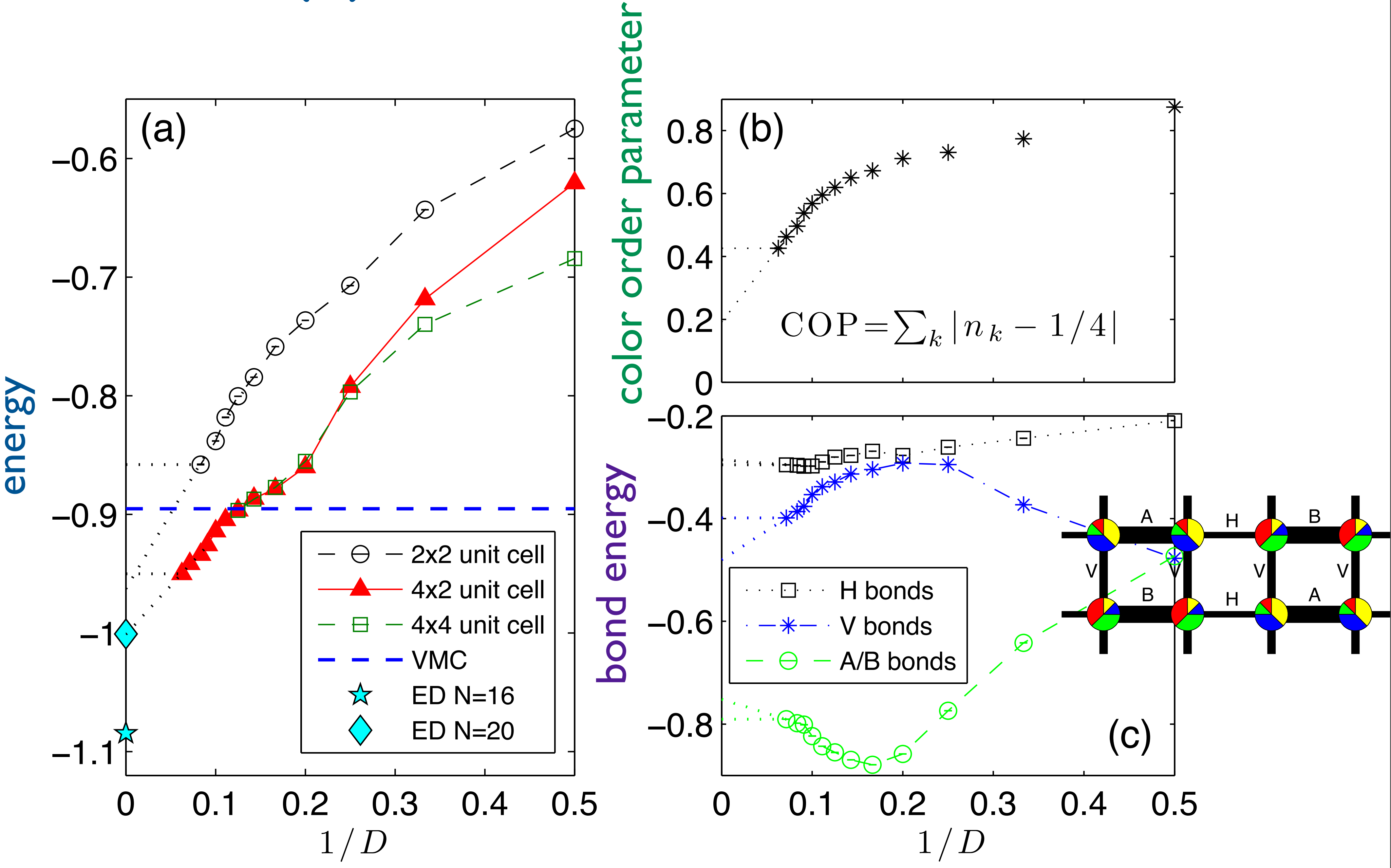
dimerization and Neel-like state:
both spatial and the SU(4)
symmetry is broken

4 × 4 = 6 + 10

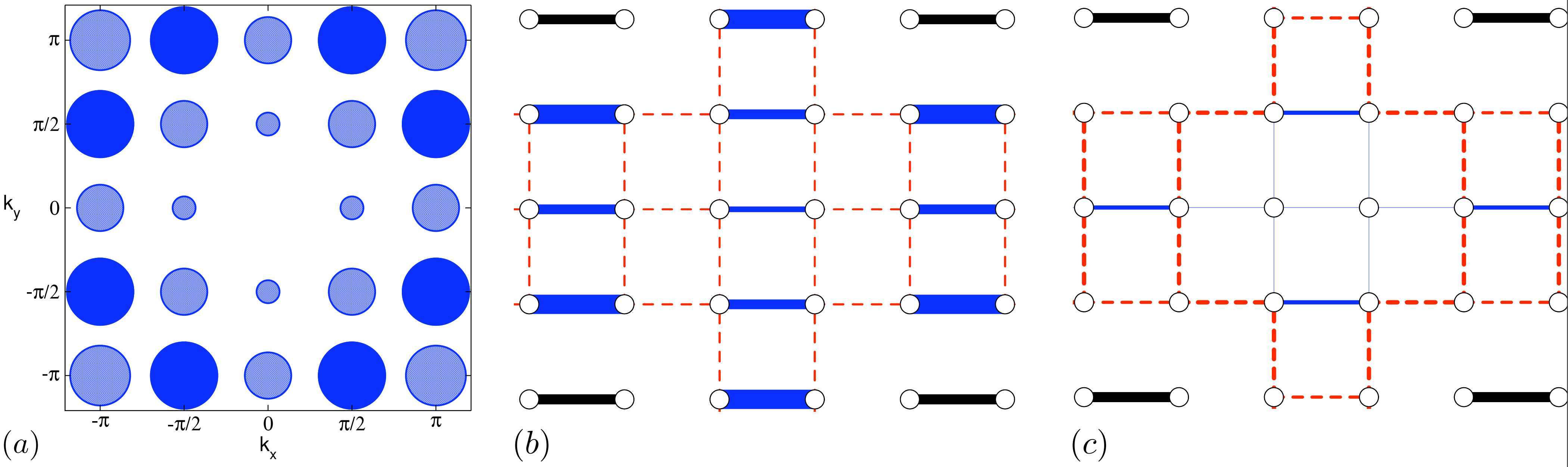


the 6 dimensional irreducible
representation is realized on the
dimers, can Neel order

SU(4) on 2D-sauare lattice: iPEPS



SU(4) on 2D-square lattice: ED



Color structure factor

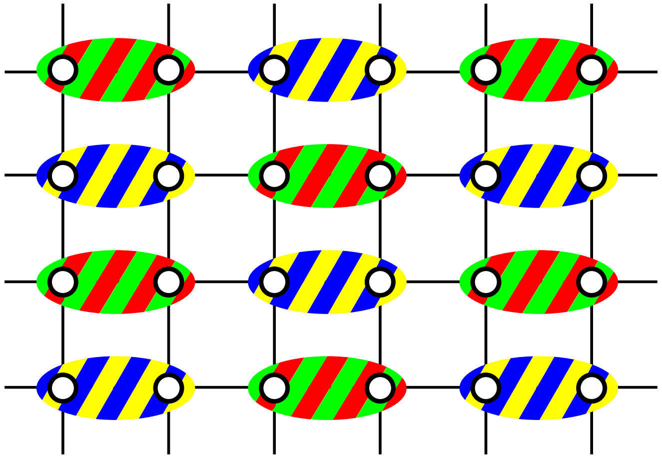
bond energy correlations

$\langle P_{ij} P_{kl} \rangle - \langle P_{ij} \rangle^2$

$\langle P_{(i,j)}(a,b) P_{(k,l)}(a,b) \rangle$

SU(4) on 2D-square lattice: our scenario

Dimerization: 6-dimensional
irreps are formed,
they can Néel order



P. Corboz, A. M. Läuchli, K. Penc, M. Troyer, F. Mila, PRL **107**, 215301 (2011).

SU(4) can be lowered to Sp(4) in cold atoms:

H. Hung, Y. Wang, and C. Wu, Phys. Rev. B **84**, 054406 (2011).
E. Szirmai and M. Lewenstein, EPL **93**, 66005 (2011).

The models

(i) SU(3) model on the triangular lattice

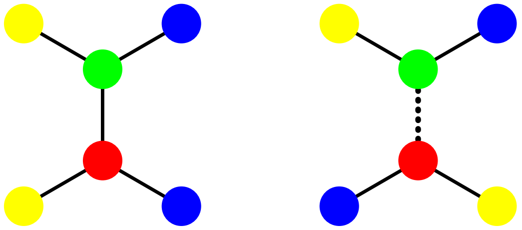
(ii) SU(3) model on the square lattice

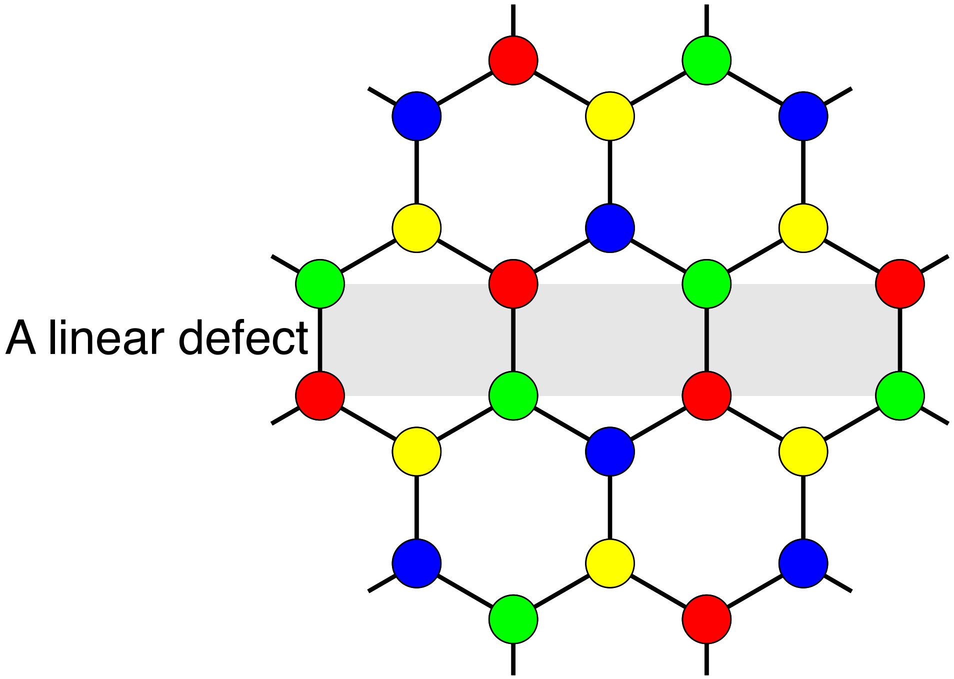
(iii) SU(4) model on the square and cubic lattice

(iv) SU(4) model on the honeycomb lattice

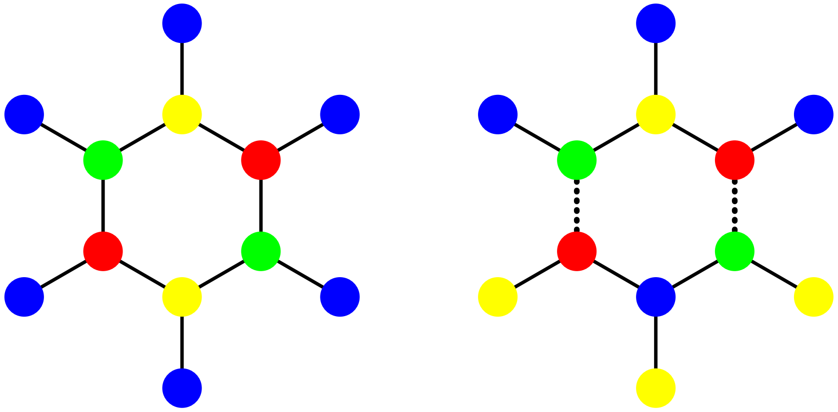
Lifting of the degeneracy in flavor wave theory

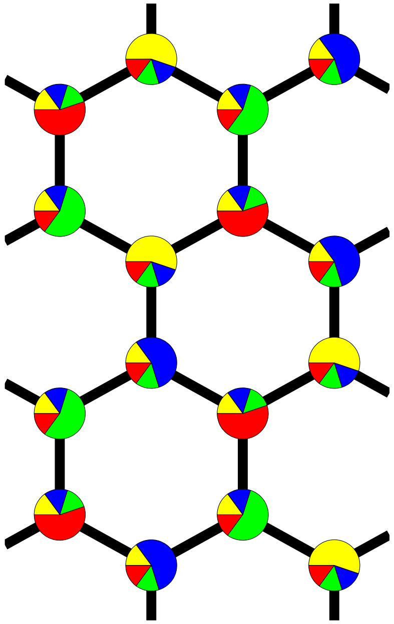
Basic building blocks:
nearest (mean field) and next nearest
(fluctuations) neighbor colors different.





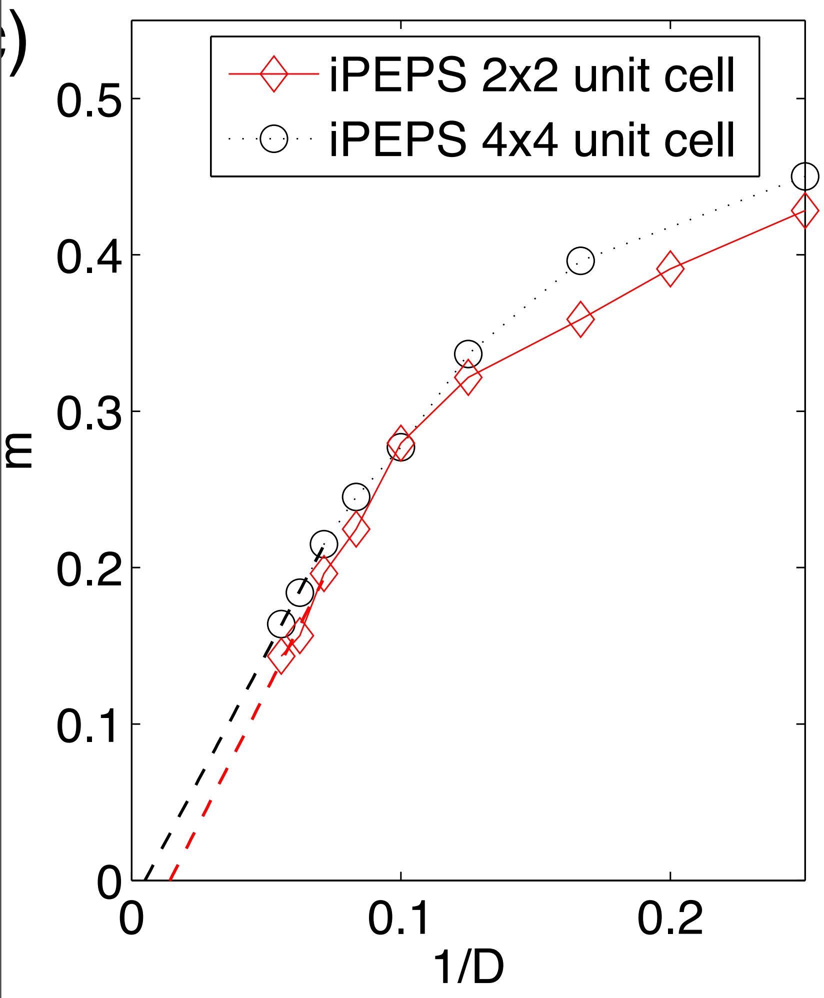
The allowed 9-spin configurations,
not lifted in harmonic approximation:

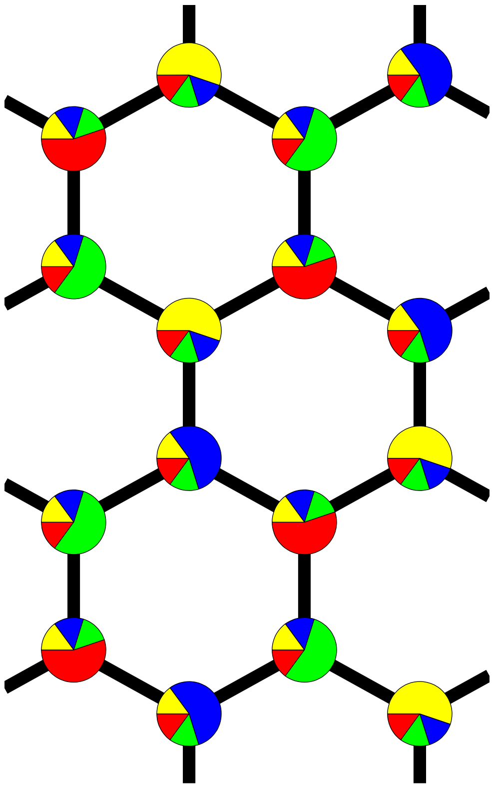




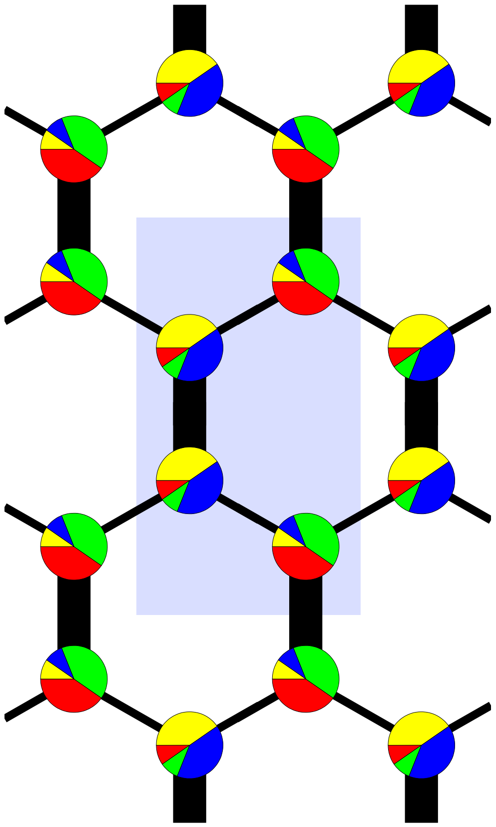
iPEPS,
D=6

iPEPS - local magnetization vanishes



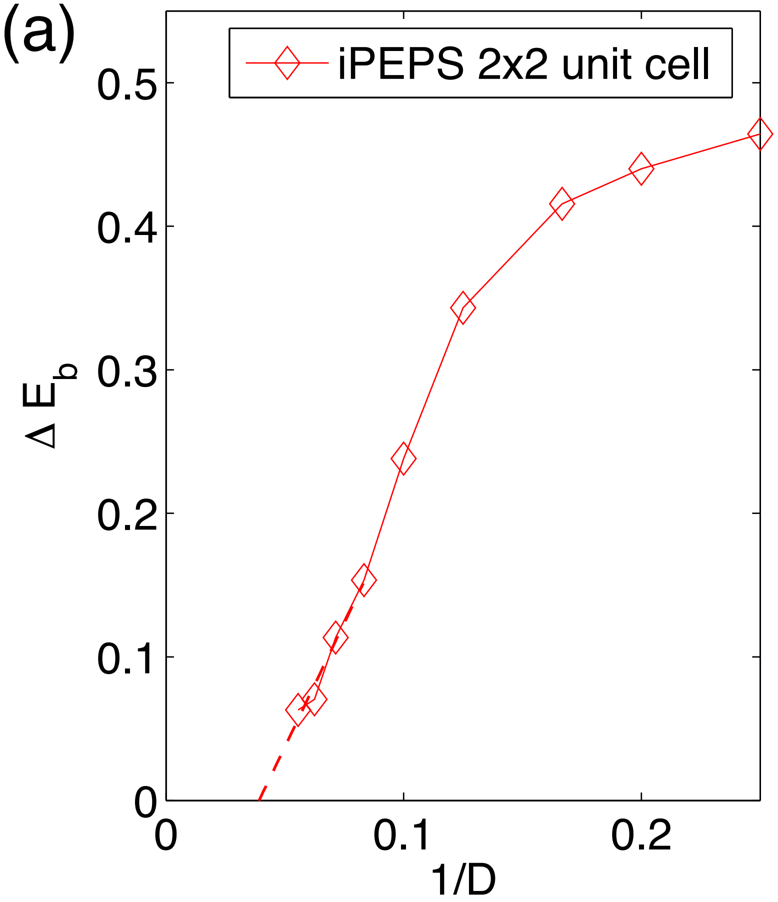


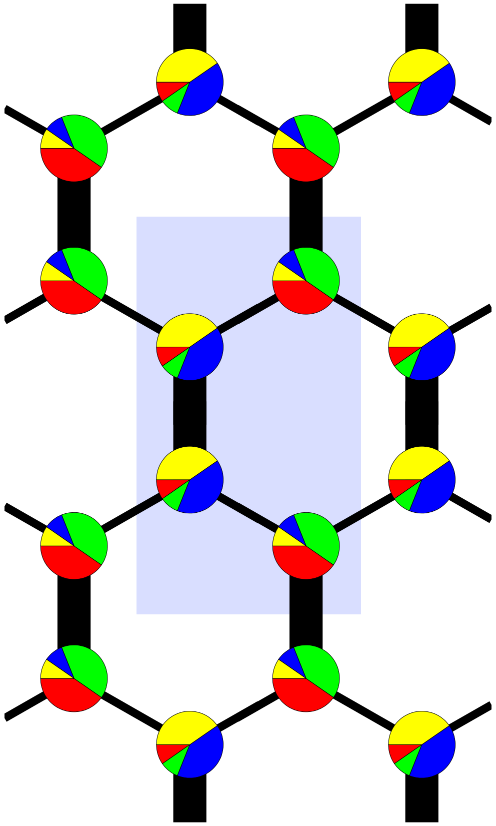
4x4 unit cell,
D=6



2x2 unit cell,
D=6

iPEPS - dimerization vanishes





2x2 unit cell,
D=6

local magnetization vanishes,
no translational symmetry breaking → flavor liquid?

we use the fermionic representation:

$$\mathcal{P}_{ij} = \sum_{\mu,\nu\in\text{colors}} f^{\dagger}_{\alpha,i} f_{\beta,i} f^{\dagger}_{\beta,j} f_{\alpha,j}$$

$$\mathcal{P}^{\text{MF}}_{ij} = \sum_{\alpha,\beta\in\text{colors}} \langle f_{\beta,i} f^{\dagger}_{\beta,j} \rangle f^{\dagger}_{\alpha,i} f_{\alpha,j}$$

$$= - \sum_{\alpha\in\text{colors}} t^{\alpha}_{ij} f^{\dagger}_{\alpha,i} f_{\alpha,j}$$

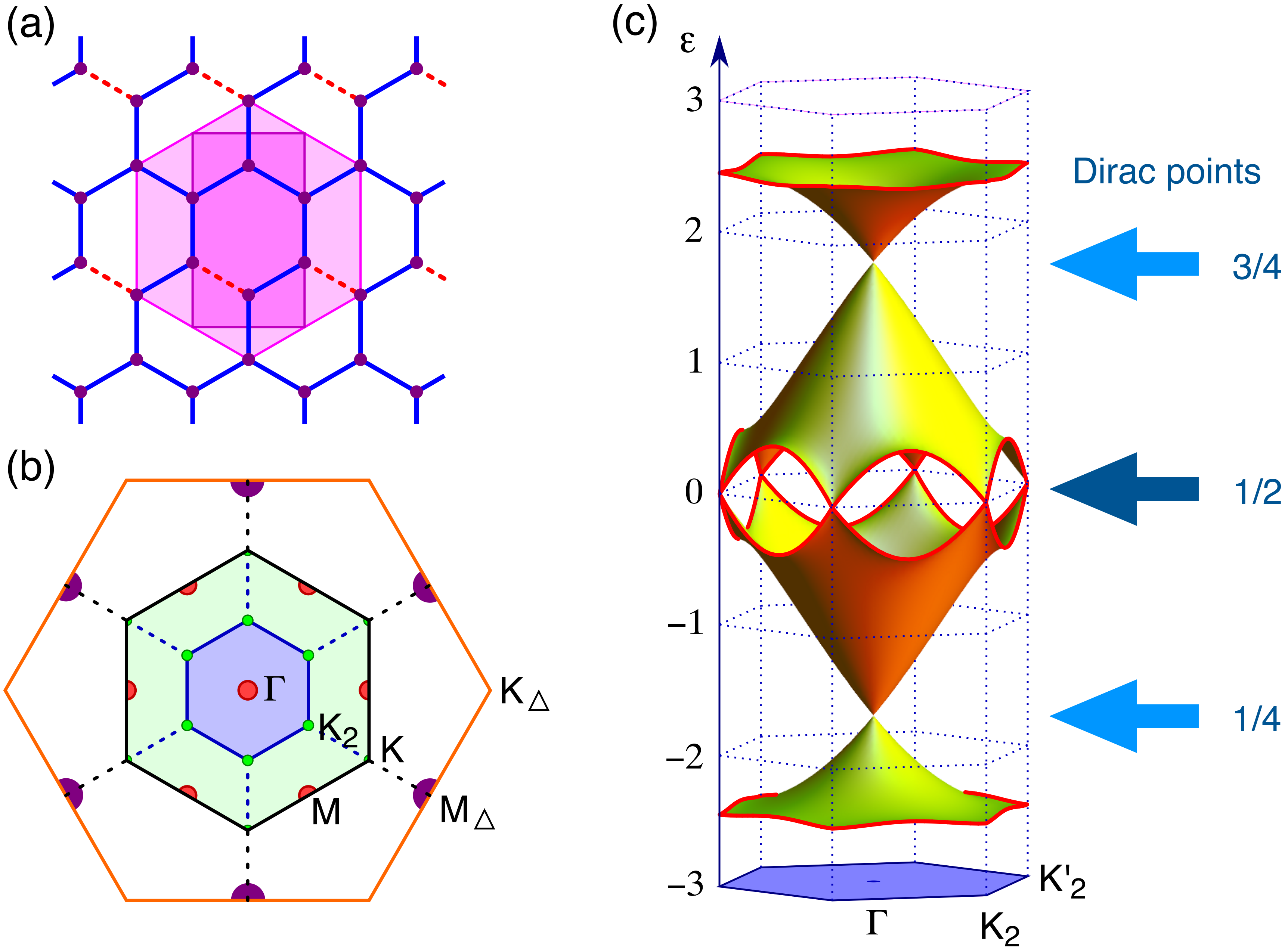
Mean-field decoupling of the fermionic Hamiltonian gives a hopping Hamiltonian and a variational wave function

$$|\Psi_{\text{vari}}\rangle = P_{\text{Gutzwiller}} |\Psi_{\text{FS}}\rangle$$

Using different Ansätze for the hoppings, we evaluate the expectation value of the Hamiltonian

$$E_{\text{vari}} = \frac{\langle \Psi_{\text{vari}} | \mathcal{H} | \Psi_{\text{vari}} \rangle}{\langle \Psi_{\text{vari}} | \Psi_{\text{vari}} \rangle}$$

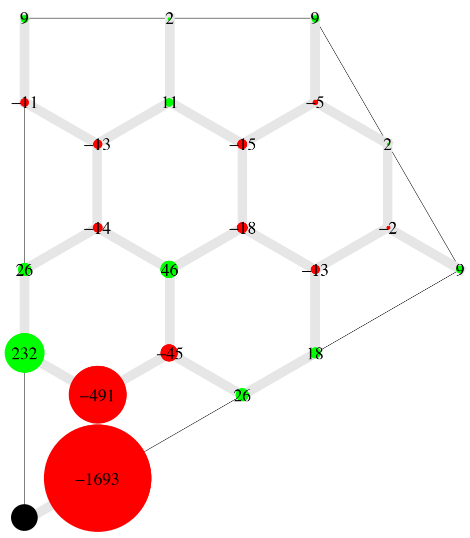
The fermionic wave function of the pi-flux state



Dirac points

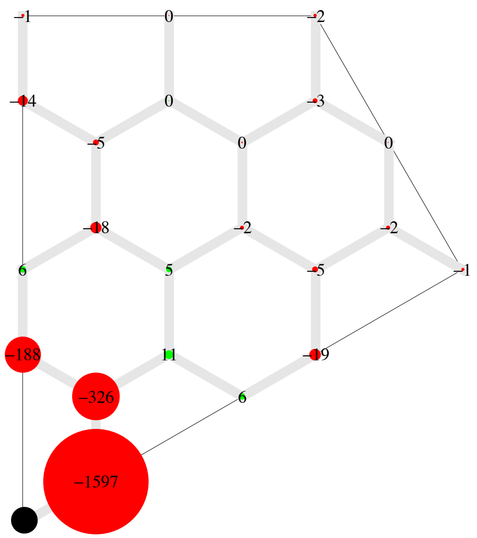
two-fold degenerate bands

96-site cluster - real space correlations from Gutzwiller projected wavefunction



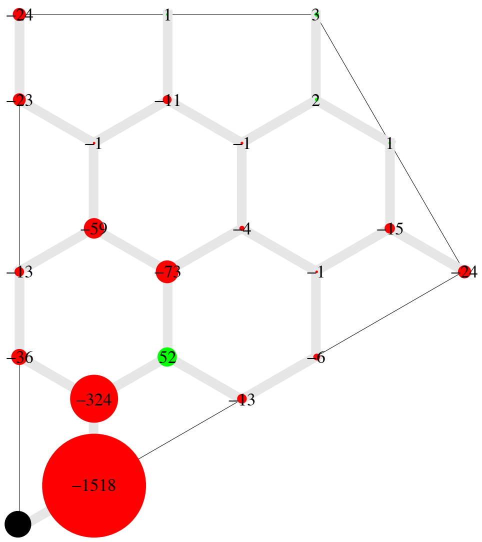
Pi-flux

E/N = –0.89466



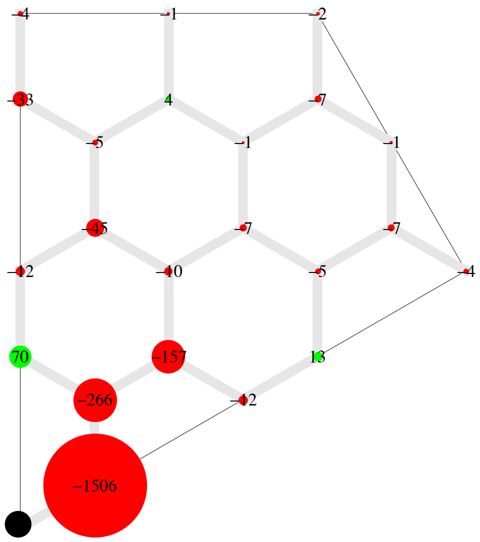
Majorana 0-flux

E/N = –0.822479



0-flux

E/N = –0.763304

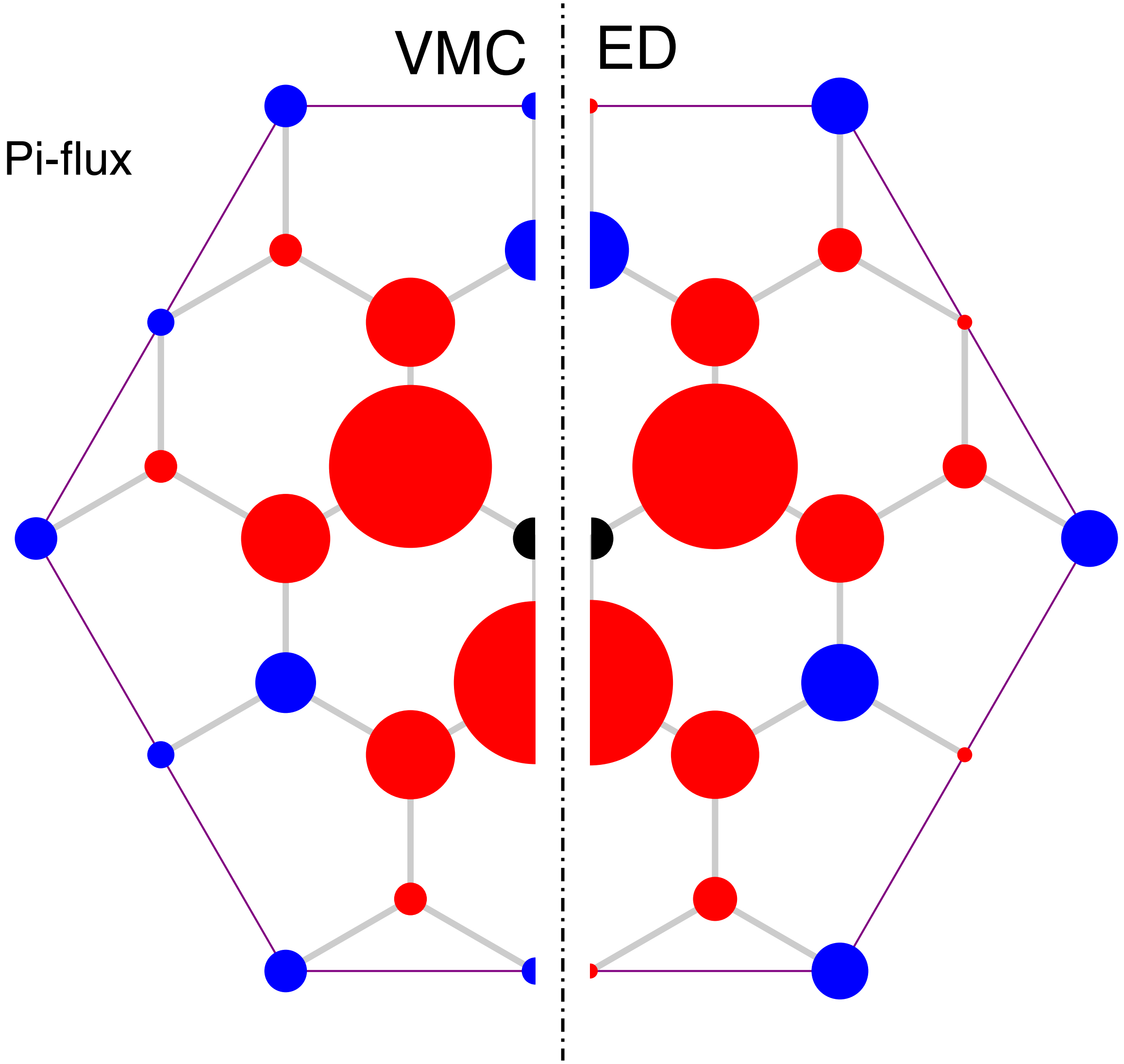


Majorana Pi-flux

E/N = –0.754662

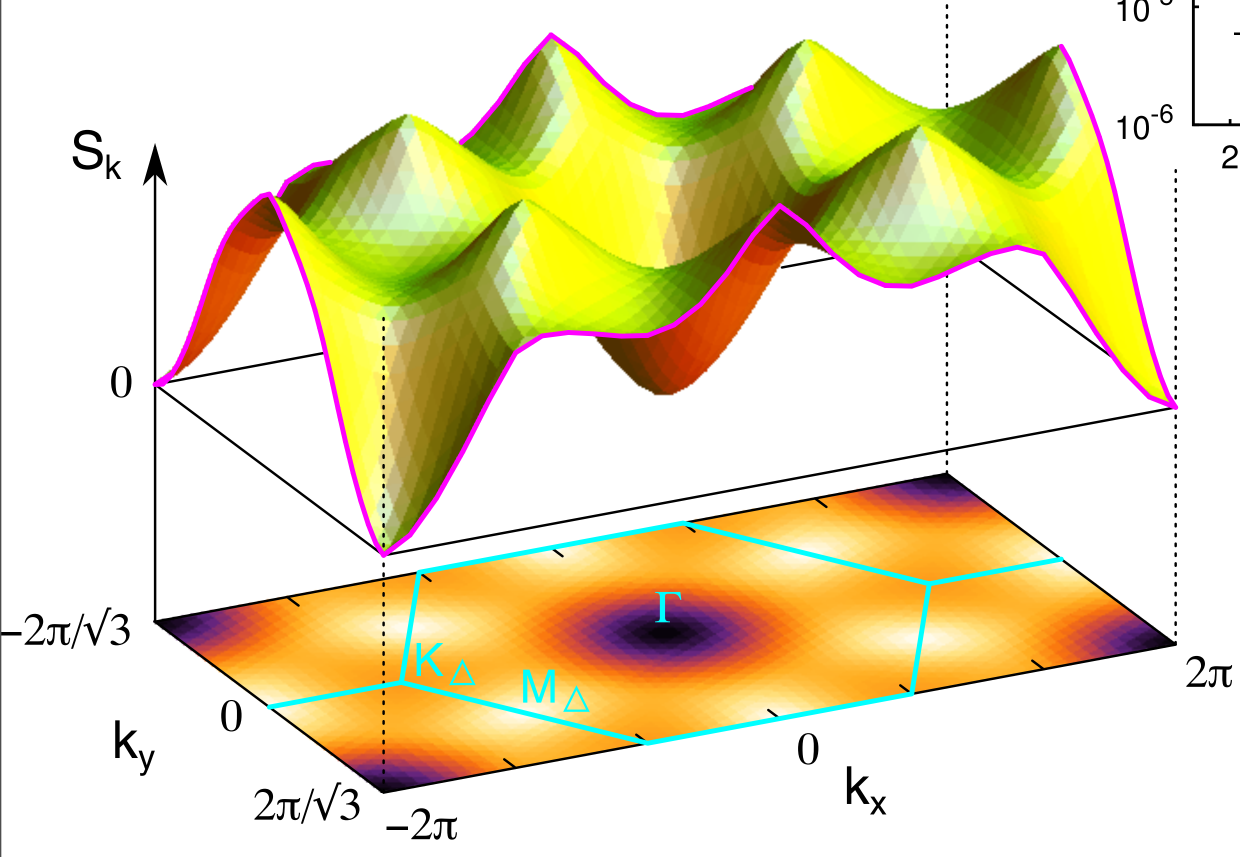
marked differences in
3rd neighbor correlations

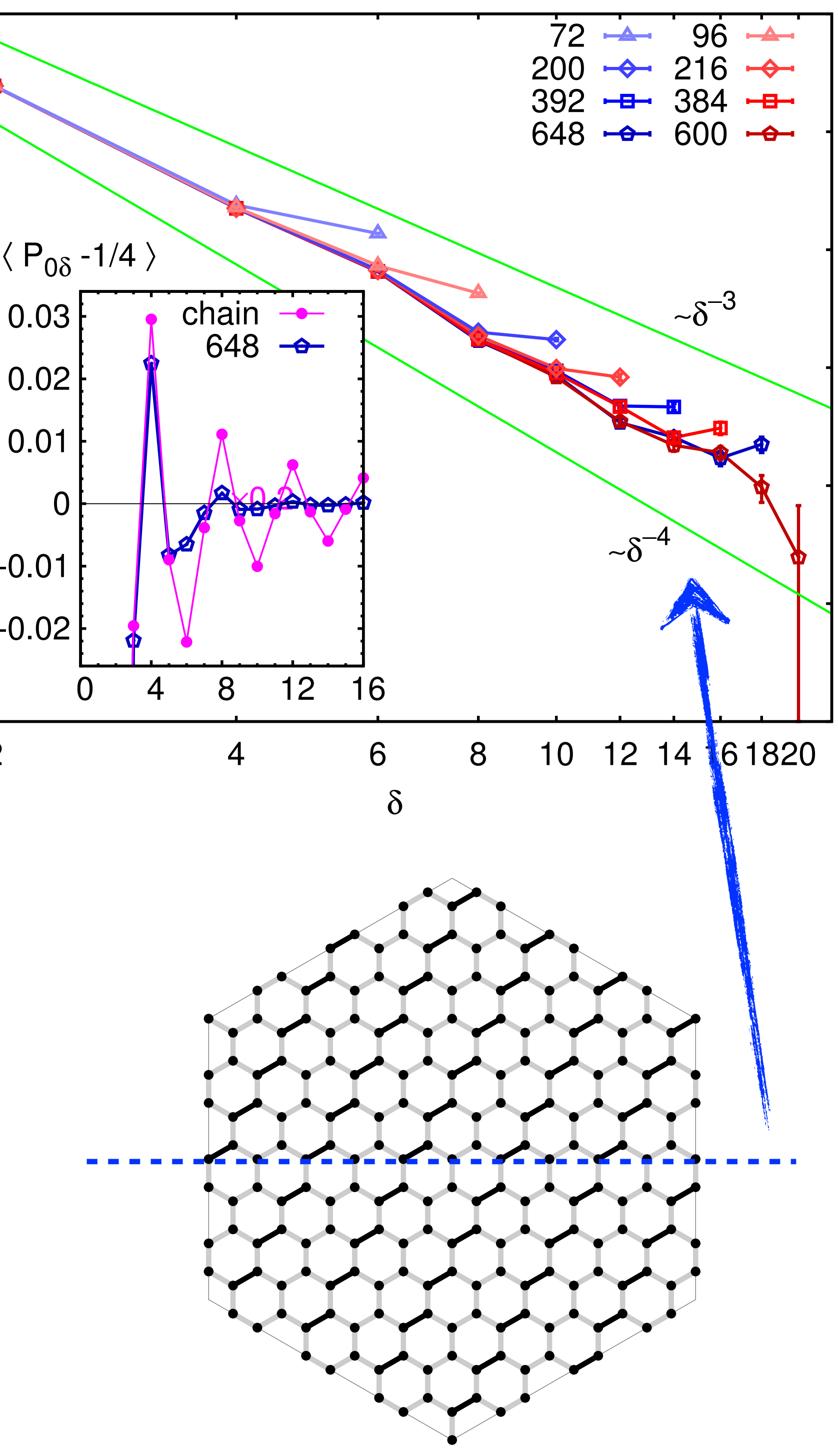
24-site cluster - real space correlations



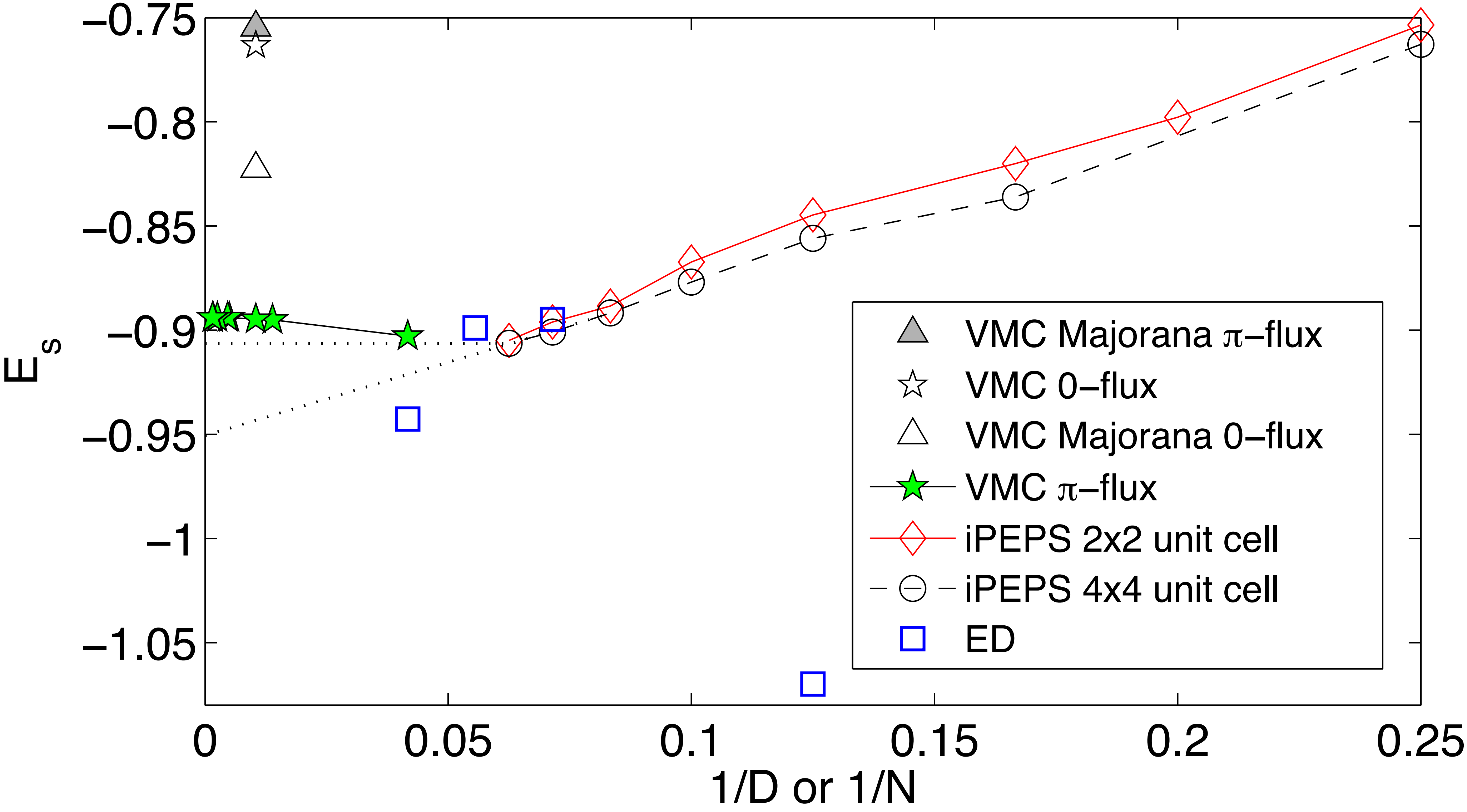
Dimension of the Hilbert space is $24!/(6!)^4$ = 2 308 743 493 056
using symmetries makes it tractable

algebraic correlations and structure factor



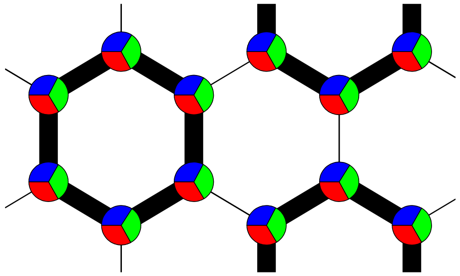


Ground state energy from different methods



SU(N) on honeycomb

SU(2) is a Néel state



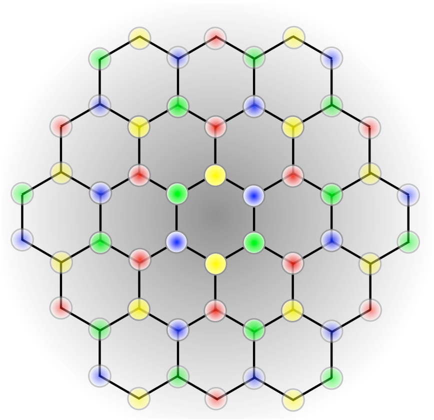
P. Corboz, unpublished

SU(3) is a plaquette state

[Y.-W. Lee and M.-F. Yang, Phys. Rev. B **85**, 100402 (2012).
H.H.Zhao,C.Xu,Q.N.Chen,Z.C.Wei,M.P.Qin,G.M. Zhang, and T. Xiang,
Phys. Rev. B **85**, 134416 (2012).]

SU(4) is most probably an algebraic flavor liquid

[P. Corboz, M. Lajkó, A. M. Läuchli, K. Penc, F. Mila, arXiv:1207.6029]



SU(6) is likely a chiral flavor liquid

[G. Szirmai E. Szirmai, A. Zamora, and M. Lewenstein, Phys. Rev. A **84**, 011611 (2011)],

similarly, SU(N) is also a chiral liquid

[extending the results of M. Hermele, V. Gurarie, & A. M. Rey, Phys. Rev. Lett. **103**, 135301 (2009)]

honeycomb optical lattices can be realized, see
poster of Johannes Hecker-Denschlag

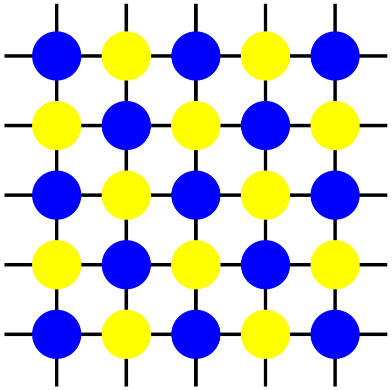
Conclusions

SU(2)

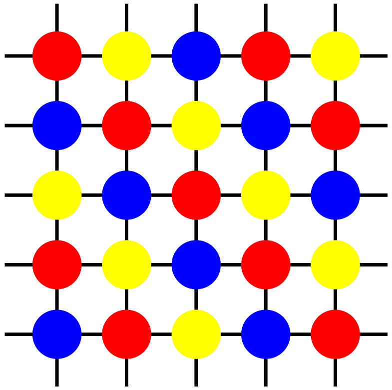
SU(3)

SU(4)

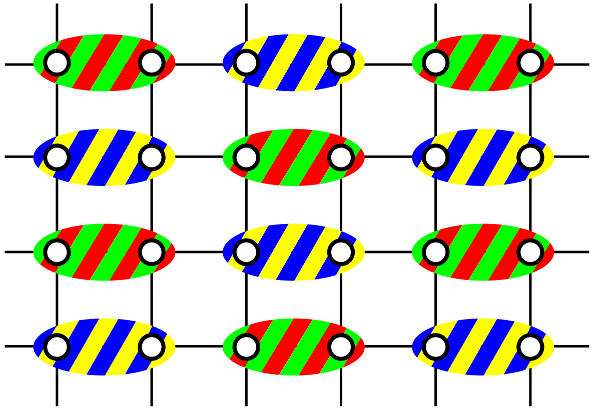
square



happy

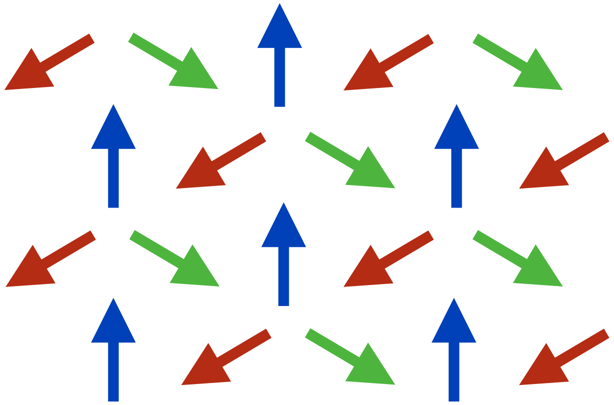


fluctuation stabilized

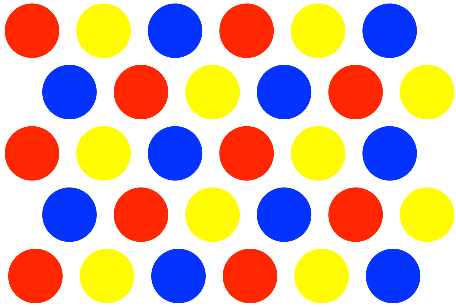


dimerization+Neel

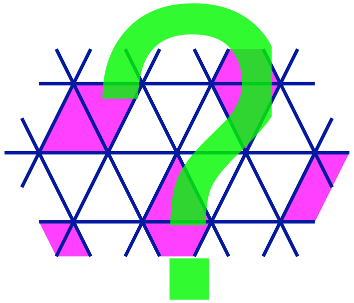
triangular



frustrated



happy



resonating
liquid

Many happy returns of the day to you, Jose!

Many happy workshops in Evora!

the end