Phase transitions in full counting statistics

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Motivation

- "Counting phase transitions" – a new way to describe correlations and characterize thermodynamic phases
- Originally motivated by "full counting statistics" (FCS) in mesoscopic physics: e.g., describing electronic noise in microcontacts
- Applicable to a wide range of systems (classical and quantum)
- For quadratic fermionic systems, related to the theory of Toeplitz determinants (Fisher–Hartwig conjecture)
- Earlier studies of phase transitions in various FCS problems: Garrahan '07, Levkivskyi and Sukhorukov '09, Karzig and von Oppen '10

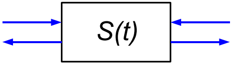
Outline

- “Counting phase transitions” in classical and quantum systems
- **Example 1:** (classical) 1D Ising model
  new “counitng phase transition”
- **Example 2:** (quantum) 1D free fermions
  subleading terms for FCS and entanglement entropy
- **Example 3:** (quantum) XY spin-1/2 chain
  new interpretation of the old phase diagram

Full counting statistics

Measuring a discrete observable:

- in time:



- or in space:

t

10001101100010100110000101

Generating function:

$$\chi_t(\lambda) = \sum_n P_n e^{i\lambda n}$$

t – duration of measurement (or length of the interval),
P_n – probability of counting n events (e.g., transmitting charge n)

Properties of the generating function

- Periodicity: $\chi_t(\lambda) = \chi_t(\lambda + 2\pi)$ – reflects charge quantization

- Multiplicativity for independent processes
("partition function"): $\chi_{A+B}(\lambda) = \chi_A(\lambda) \cdot \chi_B(\lambda)$

- In a periodic setup (with a period τ), define the extensive part:

$$\chi^{(0)}(\lambda) = \lim_{N\to\infty} \left[\chi_{N\tau}(\lambda)\right]^{1/N}$$

“Counting” phase transitions

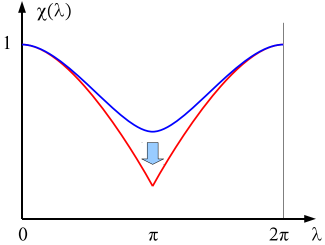
$\chi^{(0)}(\lambda)$ is periodic in λ (by construction), but:

For any t, the observable is limited
$\Rightarrow$ $\chi_t(\lambda)$ is analytic on the circle $z = e^{i\lambda}$

but

$\chi^{(0)}(\lambda)$ may develop
a singularity

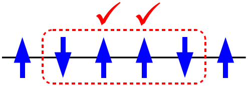
$\Rightarrow$ PHASE TRANSITION



Example 1 (classical): 1D Ising model

$$H = J\sum_j \sigma_j\sigma_{j+1} + \Gamma\sum_j \sigma_j\,, \qquad Z = \sum \exp(-H)$$

counting spins up:

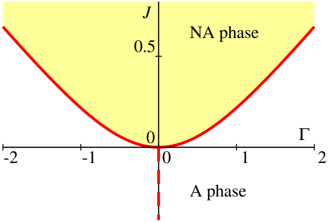


Solving by transfer matrix:

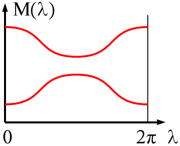
$$\begin{pmatrix}\chi^+_{N+1}(\lambda)\\ \chi^-_{N+1}(\lambda)\end{pmatrix} = M\begin{pmatrix}\chi^+_{N}(\lambda)\\ \chi^-_{N}(\lambda)\end{pmatrix}$$

$\chi^{(0)}(\lambda)$ – largest eigenvalue of the transfer matrix M

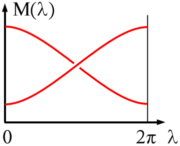
Example 1: 1D Ising model, phase diagram



Phase diagram



"Analytic" phase



"Nonanalytic" phase

Phase transition line: $\cosh \Gamma = e^{2J}$

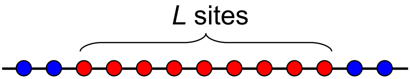
Bifurcation in the (complex) correlation length for the Jordan–Wigner string

$$V_\pi(j) = \prod_{k=j_0}^{j} \sigma_k$$

(not a phase transition in the usual sense!)

Example 2 (quantum): 1D free fermions

Free fermions on a 1D chain:



Ground state: states with momenta $|k| < k_F$ filled:

$$|\Psi\rangle = \prod_{|k|<k_F} a^+(k)\,|0\rangle$$

Generating function for counting statistics:

$$\chi_L(\lambda) = \langle\Psi|\, e^{i\lambda N_L}\,|\Psi\rangle \;, \qquad N_L = \sum_{j=1}^{L} a_j^+ a_j$$

Example 2: 1D free fermions (continued)

Average and fluctuations:

$$\langle N_L \rangle = \frac{k_F}{\pi} L \qquad \langle\langle N_L^2 \rangle\rangle \sim \frac{1}{\pi^2} \ln L$$

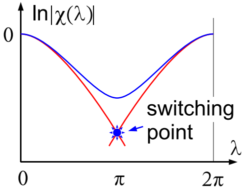
(either by bosonization or via Wick theorem)
$\Rightarrow$ fluctuations grow slower than L

At the Gaussian level (continuous bosonization):

$$\chi_L(\lambda) = \exp\left(i\lambda \langle N \rangle - \frac{\lambda^2}{2} \langle\langle N_L^2 \rangle\rangle \right) \quad \Rightarrow \quad \chi^{(0)}(\lambda) = \exp\left(i \frac{\lambda}{\pi} k_F \right)$$

– NOT periodic in λ (neglects charge discreteness)

Example 2: 1D free fermions and Toeplitz determinants



Non-analytic $\chi^{(0)}(\lambda)$:
switching branches at $\lambda = \pi$

Restoring periodicity by expressing $\chi_L(\lambda)$ as a Toeplitz determinant

$$\chi_L(\lambda) = \det \left[1 + (e^{i\lambda} - 1) \langle a_j^+ a_k \rangle \right] =: \det \sigma_{jk}$$

Fourier transform ("symbol") contains jumps at the Fermi points:

$$\sigma(q) = 1 + (e^{i\lambda} - 1)\, n_F(q)$$

— extending the Fisher–Hartwig conjecture
[Kozlowski '08, Kitanine et al '09, Deift, Its, Krasovsky '09, Calabrese, Essler '10] to obtain a full asymptotic expansion

Example 2: 1D free fermions: full asymptotic expansion

$$\chi_L(\lambda) = \sum_{n=-\infty}^{+\infty} \tilde{\chi}_L(\lambda - 2\pi n)$$

$$\tilde{\chi}_L(\lambda) = \exp\left[i\lambda \frac{k_F}{\pi} L - \frac{\lambda^2}{2\pi} \ln(2L \sin k_F) + F_0(\lambda) + F_1(\lambda, k_F) L^{-1} + \dots\right]$$

lattice version: conjectured [Calabrese, Essler '10]
continuous limit ($k_F \to 0$): verified analytically (order by order)
[DI, Abanov, Cheianov '11]

$$F_0(\lambda) = 2 \ln \left| G\left(1 + \frac{\lambda}{2\pi}\right) G\left(1 - \frac{\lambda}{2\pi}\right)\right| \qquad (G \text{ – Barnes G function})$$

$$F_1(\lambda, k_F) = -\frac{i}{4}\left(\frac{\lambda}{\pi}\right)^3 \cot k_F\,, \qquad F_2(\lambda, k_F) = \dots$$

Example 2: 1D free fermions, contnuous limit

In the continuous limit, two alternative methods:

- matrix Riemann–Hilbert problem [Cheianov, Zvonarev '03]
- Painlevé V equation [McCoy, Tang '86]

$$F_n(\lambda, k_F) = (ik_F)^{-n} f_n\left(\frac{\lambda}{2\pi}\right)$$

and all f_n are polynomials with rational coefficients:

$$f_1(\kappa) = 2\kappa^3 \qquad f_3(\kappa) = \frac{11}{2}\kappa^5 + \frac{1}{6}\kappa^3$$

$$f_2(\kappa) = \frac{5}{2}\kappa^4 \qquad f_4(\kappa) = \frac{63}{4}\kappa^6 + \frac{13}{8}\kappa^4$$

and so on (computable order by order)

Example 2: 1D free fermions, entanglement entropy

$$\rho_A = \mathrm{Tr}_B\, \rho_{\mathrm{pure}}\,, \qquad \mathcal{S} = -\mathrm{Tr}\, \rho_A \ln \rho_A$$

For free particles, spectrum of ρ_A (and hence entanglement entropy $\mathcal{S}$) follows from the full counting statitics $\chi(\lambda)$ [Klich and Levitov '09, Song et al '11, Calabrese, Mintchev and Vicari '12]

Carrying over our results from $\chi(\lambda)$ to $\mathcal{S}$ [DI, Süsstrunk '12]:

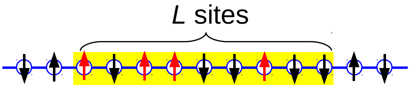
$$\mathcal{S} = \frac{1}{3}\ln(k_F L) + C + \sum_{n=1}^{\infty} s_{2n}(k_F L)^{-2n}$$

$$s_2 = -\frac{1}{12}\,, \quad s_4 = -\frac{31}{96}\,, \quad s_6 = -\frac{7057}{1440}\,, \quad \ldots$$

All s_{2n} are rational numbers computable order by order

Example 3 (quantum): XY spin-1/2 chain in a Z field

$$H = \sum_{j=-\infty}^{+\infty} \left[\left(\frac{1+\gamma}{2} \right) \sigma_j^x \sigma_{j+1}^x + \left(\frac{1-\gamma}{2} \right) \sigma_j^y \sigma_{j+1}^y - h\, \sigma_j^z \right]$$



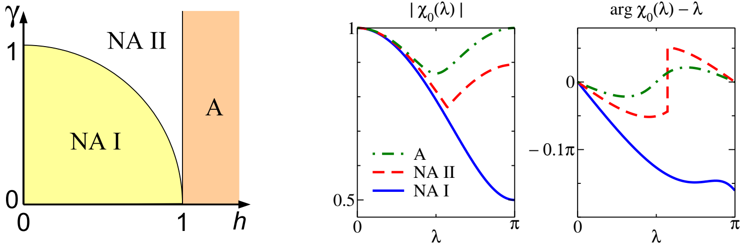
Counting up spins:

$$\chi_L(\lambda) = \langle e^{i\lambda N_L} \rangle_{T=0} \sim [\chi^{(0)}(\lambda)]^L\,, \qquad N_L = \sum_{j=1}^{L} \frac{1+\sigma_j^z}{2}$$

generalization of the free-fermion example,
mapping onto 1D quadratic BCS Hamiltonian
$\Rightarrow$ Toeplitz determinant, Szegő formula

Example 3: XY spin chain, phase diagram

Three "counting phases":

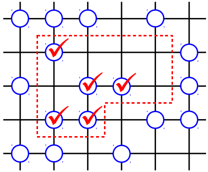


- Nonanalytic phase I: phase jump at $\lambda = \pi$ (similar to the free-fermion case)
- Nonanalytic phase II: phase jumps at intermediate λ_c
- Analytic phase: fully polarized state with local pair excitations.

— the same phase diagram in terms of spin correlations [Barouch, McCoy '71]

Summary and comments

1. "Counting phase transitions": a new way to describe correlations in terms of the analytic properties of $\chi^{(0)}(\lambda)$
2. Works for classical and quantum systems
3. Can also be applied to systems in higher dimensions:



4. Singularities in $\chi^{(0)}(\lambda)$ may occur at various values of λ [only at $\lambda=\pi$ for noninteracting fermions]
5. New results on the Fisher–Hartwig conjecture and entanglement for free 1D fermions

Summary and comments: 2

6. Physical signatures are subtle:
   - in 1D: nonanalyticity of the correlation length for Jordan–Wigner strings
   - in any D: “twisted cumulant”

$$\langle\langle n^2\rangle\rangle_\Lambda = \langle n^2\rangle_\Lambda - \langle n\rangle_\Lambda^2 = (-i\partial_\lambda)^2 \ln \chi(\lambda)|_{\lambda=\Lambda}$$

   where $\langle A\rangle_\Lambda := \langle e^{i\Lambda n} A\rangle / \langle e^{i\Lambda n}\rangle$

   $\rightarrow$ grows as t in the analytic phase and as t^2 in the nonanalytic phase