Integrable pairing models in mesoscopic physics

Jorge Dukelsky

CSIC

CONSEJO SUPERIOR
DE INVESTIGACIONES
CIENTÍFICAS

INSTITUTO DE ESTRUCTURA
DE LA MATERIA

Brief History

- Cooper pair and BCS Theory (1956-57)
- Richardson exact solution (1963).
- Gaudin magnet (1976).
- Proof of Integrability. CRS (1997).
- Recovery of the exact solution in applications to ultrasmall grains (2000).
- SU(2) Richardson-Gaudin models (2001). Rational and Hyperbolic families.
- Applications of rational RG model to superconducting grains, atomic nuclei, cold atoms, quantum dots, etc…
- Generalized RG Models for r>1 (2006-2009). SO(6) Color pairing . SO(5) T=1 and SO(8) T=0,1 p-n pairing model and spin 3/2 cold atoms.
- Realization of the hyperbolic family in terms of a p-wave integrable pairing Hamiltonian (2010). Applications to nuclear structure (2011).

Richardson's Exact Solution

ume 3, number 6 PHYSICS LETTERS 1 February 1963

A RESTRICTED CLASS OF EXACT EIGENSTATES OF THE PAIRING-FORCE HAMILTONIAN *

R. W. RICHARDSON
H. M. Randall Laboratory of Physics,
University of Michigan, Ann Arbor, Michigan

Received 23 November 1962

Exact Solution of the BCS Model

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

Eigenvalue equation:

$$H_P |\Psi\rangle = E |\Psi\rangle$$

Ansatz for the eigenstates (generalized Cooper ansatz)

$$|\Psi\rangle = \prod_{\alpha=1}^{M} \Gamma^{\dagger}_{\alpha} |0\rangle, \quad \Gamma^{\dagger}_{\alpha} = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c^{+}_{k\uparrow} c^{+}_{-k\downarrow}$$

Richardson equations

$$1+g\sum_{k=0}\frac{1}{2\varepsilon_k-E_\alpha}+2g\sum_{\beta(\neq\alpha)=1}^{M}\frac{1}{E_\alpha-E_\beta}=0,\qquad E=\sum_{\alpha=1}^{M}E_\alpha$$

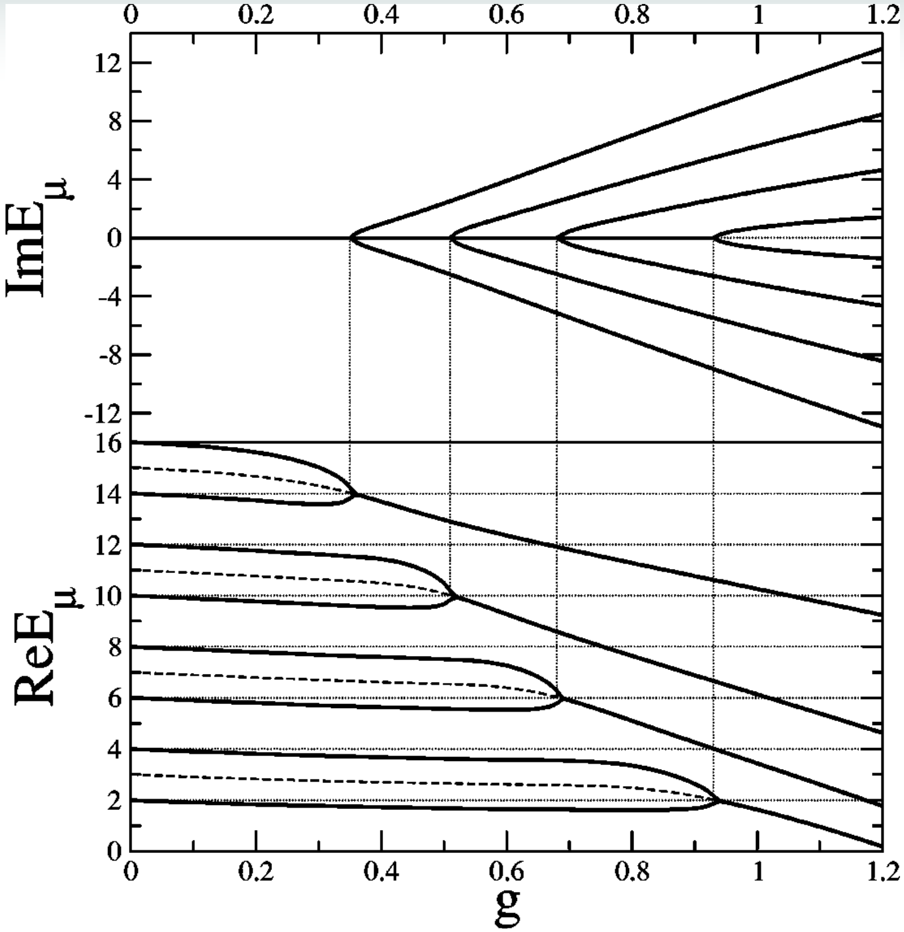
Properties:

This is a set of M nonlinear coupled equations with M unknowns (E_α).

The pair energies are either real or complex conjugated pairs.

There are as many independent solutions as the dimension of the Hilbert space. The solutions can be classified in the weak coupling limit ($g\to 0$).

Exact solvability reduces an exponential complexity of the many-body problem to an algebraic problem.



Evolution of the real and imaginary par of the pair energies with *g.*
L=16, M=8

Integrals of motion of the Richardson-Gaudin Models

L. Amico, A. Di Lorenzo, and A. Osterloh , Phys. Rev. Lett. 86, 5759(2001)
J. D., C. Esebbag and P. Schuck, Phys. Rev. Lett. 87, 066403 (2001).

- Pair realization of the SU(2) algebra

$$S_j^z = \frac{1}{2}\sum_m a_{jm}^+ a_{jm} - \frac{\Omega_j}{4} \; , \; S_j^+ = \frac{1}{2}\sum_m a_{jm}^+ a^+_{j\bar{m}}$$

- The most general combination of linear and quadratic generators, with the restriction of being hermitian and number conserving, is

$$R_i = S_i^z + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2}\left(S_i^+ S_j^- + S_i^- S_j^+\right) + Z_{ij} S_i^z S_j^z \right]$$

- The integrability condition $\left[R_i, R_j\right] = 0$ leads to

$$Z_{ij} X_{jk} + X_{jk} Z_{ki} + X_{ki} X_{ij} = 0$$

•These are the same conditions encountered by Gaudin (J. de Phys. 37 (1976) 1087) in a spin model known as the Gaudin magnet.

Gaudin (1976) found three solutions

XXX (Rational)

$$X_{ij} = Z_{ij} = \frac{1}{\eta_i - \eta_j}$$

XXZ (Hyperbolic ≡ Trigonometric)

$$X_{ij} = \frac{1}{Sinh\left(x_i - x_j\right)} = 2\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Z_{ij} = Coth\left(x_i - x_j\right) = \frac{\eta_i + \eta_j}{\eta_i - \eta_j}$$

Exact solution

$$R_i \left|\Psi\right\rangle = r_i \left|\Psi\right\rangle$$

Eigenstates of the Rational Model : Richardson Ansatz

$$\left|\Psi_{XXX}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{1}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle, \quad \left|\Psi_{XXZ}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle$$

- Any function of the *R* operators defines a valid integrable and exactly solvable Hamiltonian..

- Within the pair representation two body Hamiltonians can be obtained by a linear combination of *R* operators:

$$H = \sum_{l=1}^{L} \varepsilon_l R_l(\eta, g) + C$$

- The parameters *g*, η´s and ε´s are arbitrary. There are 2 *L*+1 free parameters to define an integrable Hamiltonian in each of the families. (*L* number of single particle levels)

- The BCS Hamiltonian solved by Richardson can be obtained from the *XXX* family by choosing $\eta = \varepsilon$.

$$H_{BCS} = \sum_i 2\varepsilon_i S_i^z + g \sum_{ij} S_i^+ S_j^-$$

- An important difference between RG models and any other ES model is the large number of free parameters. They can be used to define physical interactions. They can even be chosen randomly.

PHYSICAL REVIEW E **70**, 026208 (2004)

Stringent numerical test of the Poisson distribution for finite quantum integrable Hamiltonians

A. Relaño,[1] J. Dukelsky,[2] J. M. G. Gómez,[1] and J. Retamosa[1]

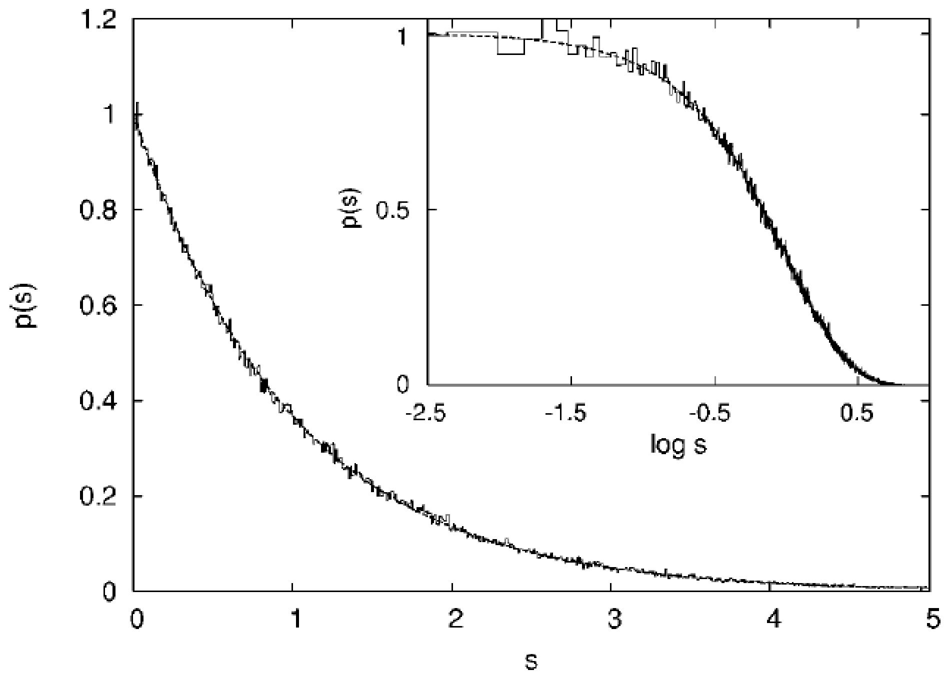


FIG. 2. Nearest-neighbor spacing distribution $P(s)$ for 200 TB-PRE members. The dashed curve corresponds to the Poisson limit.

200 random ensemble

$$L = 13$$

$$M = 6$$

$$D = 1716$$

Some models derived from **rational (XXX)** RG

- BCS Hamiltonian (Fermion and Boson).
- Generalized Pairing Hamiltonians (Fermion and Bosons).
- The Universal Hamiltonian of quantum dots.
- Central Spin Model.
- Generalized Gaudin magnets.
- Lipkin Model.
- Two-level boson models (IBM, molecular, etc..)
- Atom-molecule Hamiltonians (Feshbach resonances), or Generalized Jaynes-Cummings models,
- Breached superconductivity (Sarma state).
- Pairs with finite center of mass momentum, FFLO superconductivity.

Review: J.Dukelsky, S. Pittel and G. Sierra, Rev. Mod. Phys. 76, 643 (2004).

The Hyperbolic Richardson-Gaudin Model

A particular RG realization of the hyperbolic family is the separable pairing Hamiltonian:

$$H = \sum_i \eta_i R_i = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-$$

With eigenstates:

$$|\Phi_M\rangle = \prod_{\alpha=1}^{M} \left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+ \right) |0\rangle, \qquad E(\Phi_M) = \sum_{\alpha=1}^{M} E_\alpha$$

Richardson equations:

$$0 = \sum_i \frac{s_i}{\eta_i - E_\alpha} - \frac{Q}{E_\alpha} + \sum_{\alpha'(\neq\alpha)} \frac{1}{E_\alpha - E_{\alpha'}}, \quad 2Q = \frac{1}{G} - L + M - 2$$

The physics of the model is encoded in the exact solution. It does not depend on any particular representation of the Lie algebra

(p_x+ip_y) exactly solvable model

In 2D one can find a representation of the SU(2) algebra in terms of spinless fermions.

$$S_k^z = \frac{1}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k} - 1\right), \qquad S_k^+ = \frac{k_x + ik_y}{|k|} c_k^\dagger c_{-k}^\dagger = \left(S_k^-\right)^\dagger$$

Choosing $\eta_k = k^2$ we arrive to the p_x+ip_y Hamiltonian

$$H = \sum_{k(k_x>0)} \frac{k^2}{2}\left(c_k^\dagger c_k + c_{-k}^\dagger c_{-k}\right) - G \sum_{\substack{k,k' \\ (k_x,k_x'>0)}} \left(k_x + ik_y\right)\left(k_x - ik_y\right) c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}$$

M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao, Phys. Rev. B 79, 180501 (2009).
C. Dunning, M. I. Ibañez, J. Links, G. Sierra and S. Y. Zhao,, J. Stat. Mech. P080025 (2010).
S. Rombouts, J. Dukelsky and G. Ortiz, Phys. Rev. B. 82, 224510 (2010).

Why *p*-wave pairing?

- $p_x + ip_y$ paired phase has been proposed to describe the A1 superfluid phase of ^{3}He.

- N. Read and D. Green (Phys. Rev. B 61, 10267 (2000)), studied the $p_x + ip_y$ model. They showed that *p*-wave pairing has a QPT (2º order?) separating two gapped phases: a) a non-trivial topological phase. **Weak pairing**; b) a phase characterized by tightly bound pairs. **Strong pairing**.

- Moreover, there is a particular state in the phase diagram (the Moore-Read Pfafian) isomorphic to the ν=5/2 fractional quantum Hall state.

- In polarized (single hyperfine state) cold atoms *p*-wave pairing is the most important scattering channel (*s*-wave is suppressed by Pauli). *p*-wave Feshbach resonances have been identified and studied. However, a *p*-wave atomic superfluid is unstable due to atom-molecule and molecule-molecule relaxation processes.

- Current efforts to overcome these difficulties. The great advantage is that the complete BCS-BEC transition could be explored.

From the exact solution

1) The Cooper pair wavefunction

$$\Gamma_\alpha^+ = \sum_k \frac{k_x + ik_y}{k^2 - E_\alpha} c_k^+ c_{-k}^+$$

- E_α real positive $\rightarrow$ uncorrelated pair
- E_α complex $\rightarrow$ Cooper Resonance
- E_α real negative $\rightarrow$ Bound state

2) All pair energies converge to zero (Moore-Read line)

$$G = \frac{1}{L - M + 1}, \qquad g = \frac{1}{1-\rho} \Rightarrow E = 0\,;$$

$$\left|\Phi_M\right\rangle_{Exact} = \left[\sum_{k,k_x>0} \frac{1}{k_x - ik_y} c_k^\dagger c_{-k}^\dagger\right]^M \left|0\right\rangle = \left|\Phi_M\right\rangle_{PBCS}$$

Density $\rho = M / L$
Coupling $g = GL$

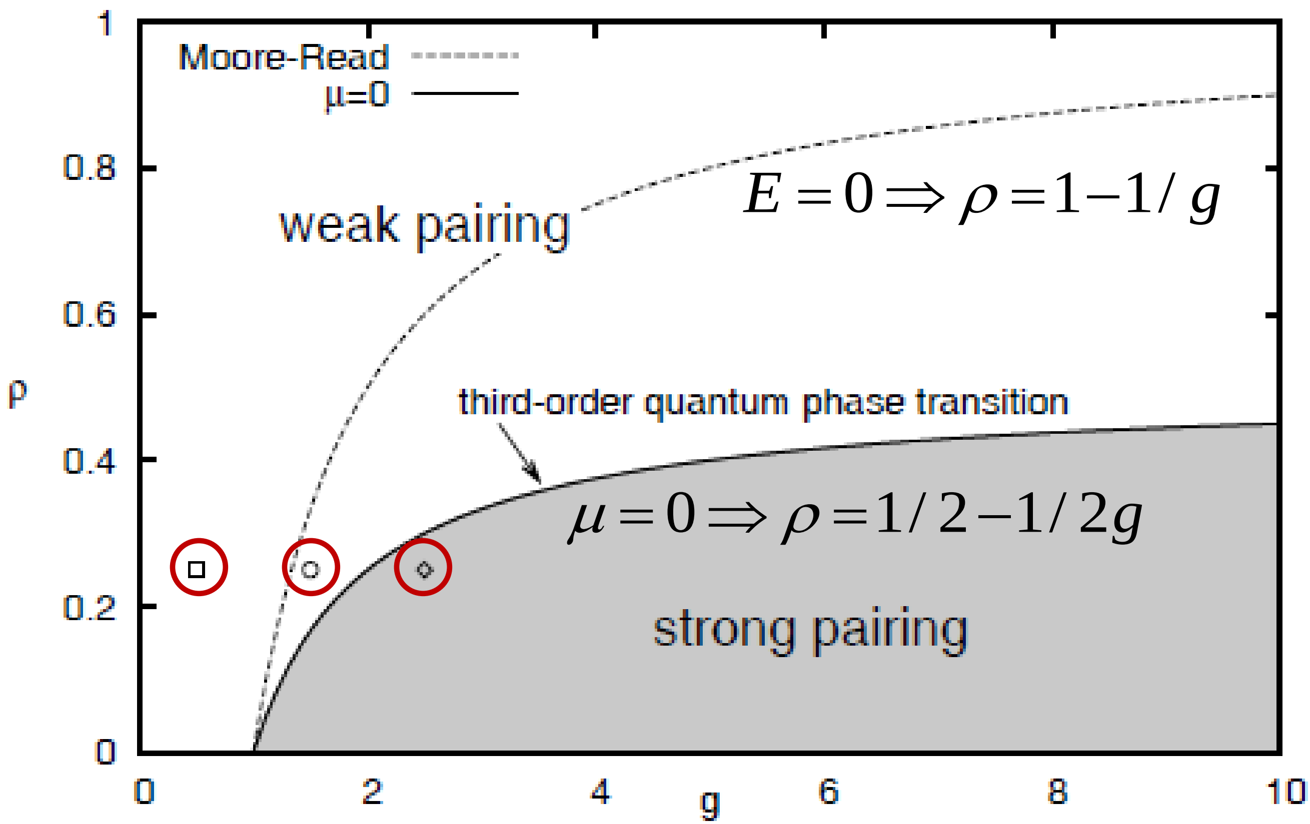
3) All pair energies real and negative (Phase transition)

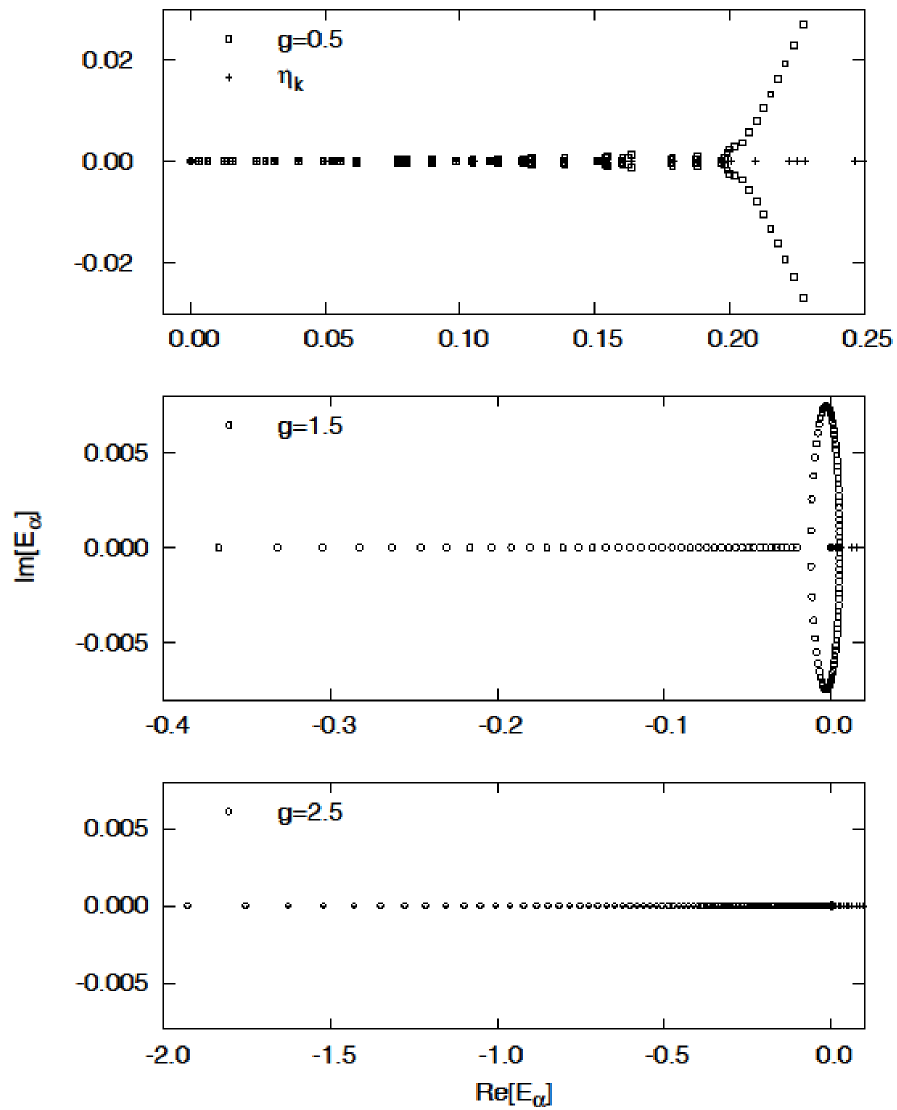
$$G \geq \frac{1}{L - 2M + 1}, \quad g \geq \frac{1}{1-2\rho}$$

$$for \quad g = 1/\left(1-2\rho\right) \Rightarrow E_1 = 0$$

Quantum phase diagram of the hyperbolic model

The phase diagram can be parametrized in terms of the density $\rho = M / L$
and the rescaled coupling $g = GL$
In the thermodynamic limit the Richardson equations $\longrightarrow$ BCS equations





Exact solution in a 2D lattice with disk geometry of R=18 with total number of levels L=504 and M=126. (quarter filling)

$D \cong 10^{122}$

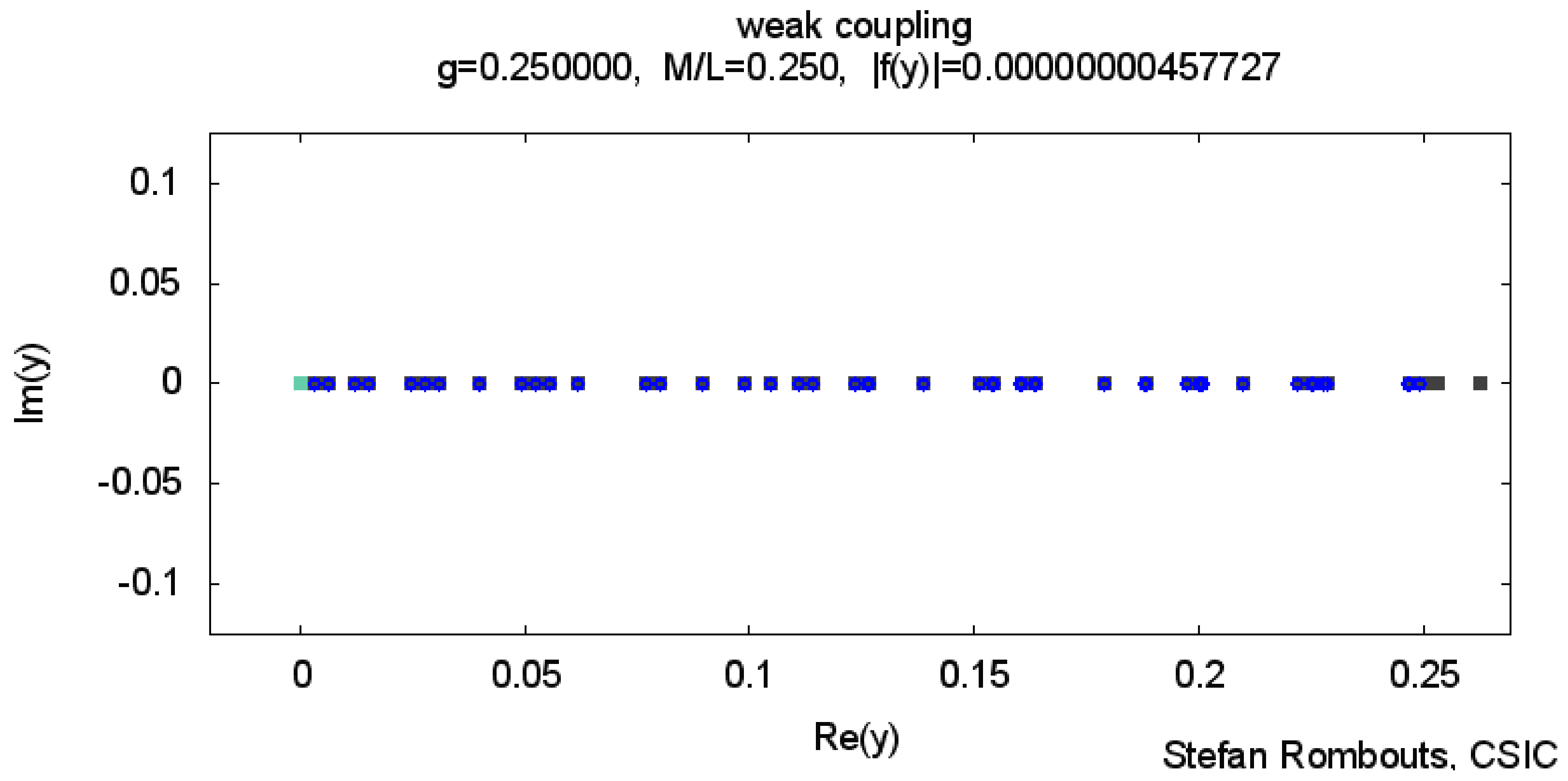
g=0.5 weak pairing

g=1.33 Moore-Read

g=1.5 weak pairing

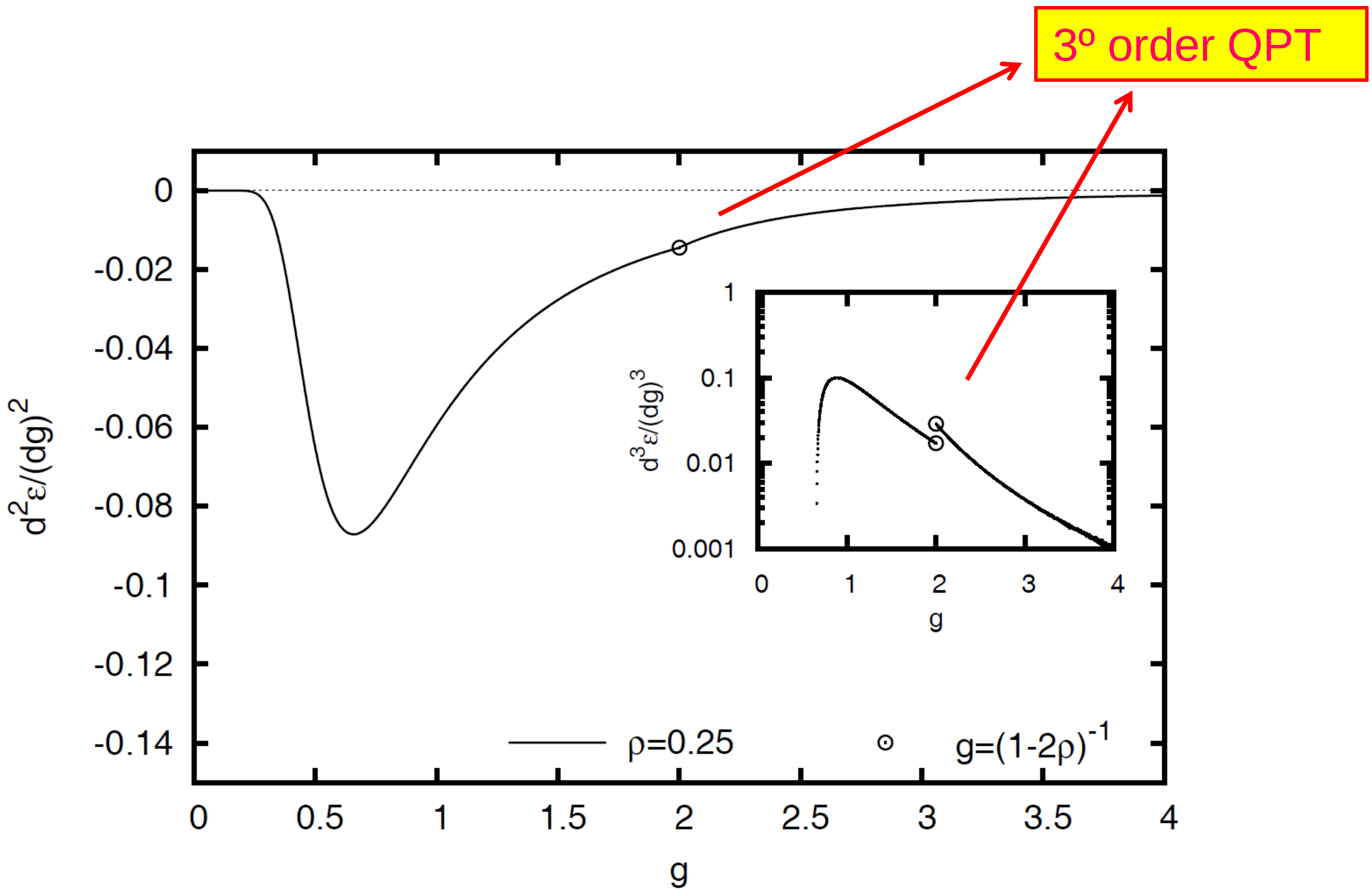
g=2.0 QPT

g=2.5 strong pairing



weak coupling
g=0.250000, M/L=0.250, |f(y)|=0.00000000457727
0.1
0.05
0
-0.05
-0.1
Im(y)
0
0.05
0.1
0.15
0.2
0.25
Re(y)
Stefan Rombouts, CSIC

Higher order derivatives of the GS energy in the thermodynamic limit



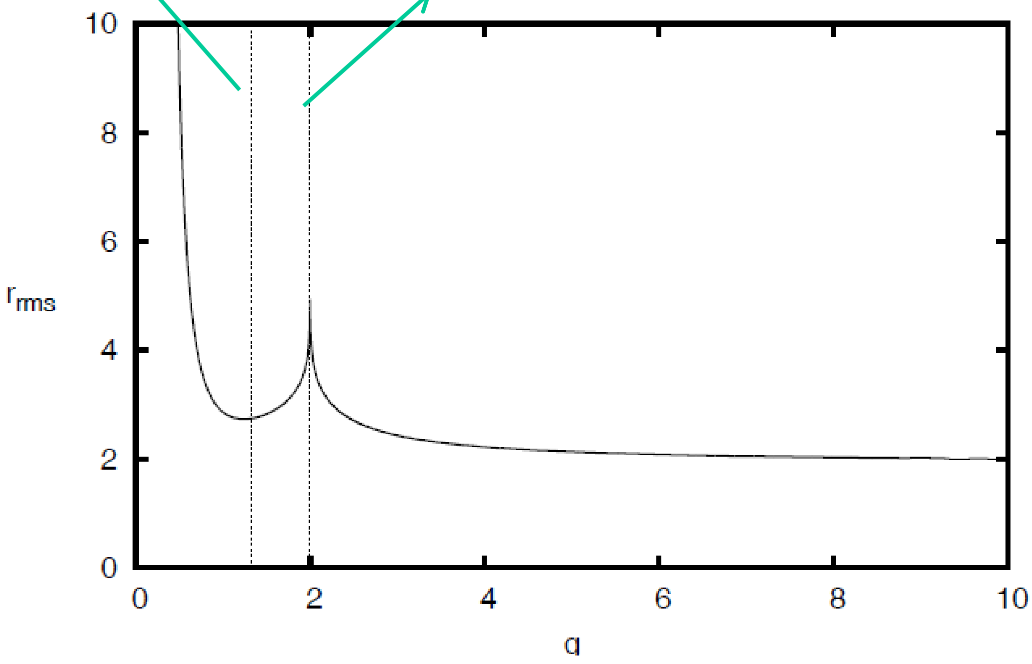
Characterization of the QPT

In the thermodynamic limit the condensate wavefunction in k-space is:

$$\phi(k) = \langle \psi | c_k^\dagger c_{-k}^\dagger | \psi \rangle = u_k v_k$$

Moore-Read

QPT, $\mu = 0$



Size of the pair wavefunction

$$r_{rms}^2 = \frac{\int |\nabla \phi(k)|^2 dk}{\int |\phi(k)|^2 dk}$$

$$\lim_{\mu \to 0} \left(r_{rms}^2 \right) \approx Ln|\mu|$$

Accessible experimentally by quantum noise interferometry and time of flight analysis?

A similar analysis can be applied to the pairs in the exact solution

$$S^{+}\left(E_{\alpha}\right)=\sum_{k}\phi_{\alpha}\left(k\right)c_{k}^{\dagger}c_{-k}^{\dagger}, \quad \phi_{\alpha}\left(k\right)\propto\frac{k_{x}+ik_{y}}{k^{2}-E_{\alpha}}$$

The root mean square $r_{rms,exact}^{2}$ of the pair wavefunction is finite for *E* complex or real and negative.

However, $r_{rms,exact}^{2}\Rightarrow\infty$ for $E\ real\ and\geq 0$

In strong pairing all pairs are bound and have finite radius.

At the QPT one pair energy becomes real an positive corresponding to a single deconfined Cooper pair on top of an ensemble of bound molecules.

The Hyperbolic Model in Nuclear Structure

J. Dukelsky, S. Lerma H., L. M. Robledo, R. Rodriguez-Guzman, S. Rombouts, PRC (in press)

The separable integrable Hyperbolic Hamiltonian

$$H = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j}\, S_i^+ S_j^-$$

Redefining the 0 of energy $\eta_i = \varepsilon_i - \alpha$, absorbing the constant in the chemical potential μ

Exactly solvable H with non-constant matrix elements

$$H = \sum_i (\varepsilon_i - \mu) c_i^+ c_i - G \sum_{i,j} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)}\, c_i^+ c_{\bar{i}}^+ c_{\bar{j}} c_j$$

α is a new parameter that could serve as an energy cutoff.

In BCS approximation:

$$\Delta_i = G\sqrt{\alpha - \varepsilon_i} \sum_{i'} \sqrt{\alpha - \varepsilon_{i'}}\, u_{i'} v_{i'} = \Delta \sqrt{\alpha - \varepsilon_i}$$

The BCS Hamiltonian has

$$\Delta_i = \Delta$$

Gogny force in nuclear physics

- **Phenomenological effective density dependent force.**
- **The central part is finite range, providing a natural cutoff for pairing correlations**
- **The parameters were adjusted to nuclear matter properties, to some selected finite nuclei, and to the pairing properties in Tin isotopes.**
- **It has been used over the years to reproduce and explain the low energy properties in atomic nuclei in the whole table o nuclides.**

$$\mathrm{V}_{12} = \sum_{j=1}^{2} e^{-(\vec{r}_1-\vec{r}_2)^2/\mu_j{}^2} \left(W_j + B_j P_\sigma - H_j P_\tau - M_j P_\sigma P_\tau \right)$$ Central part

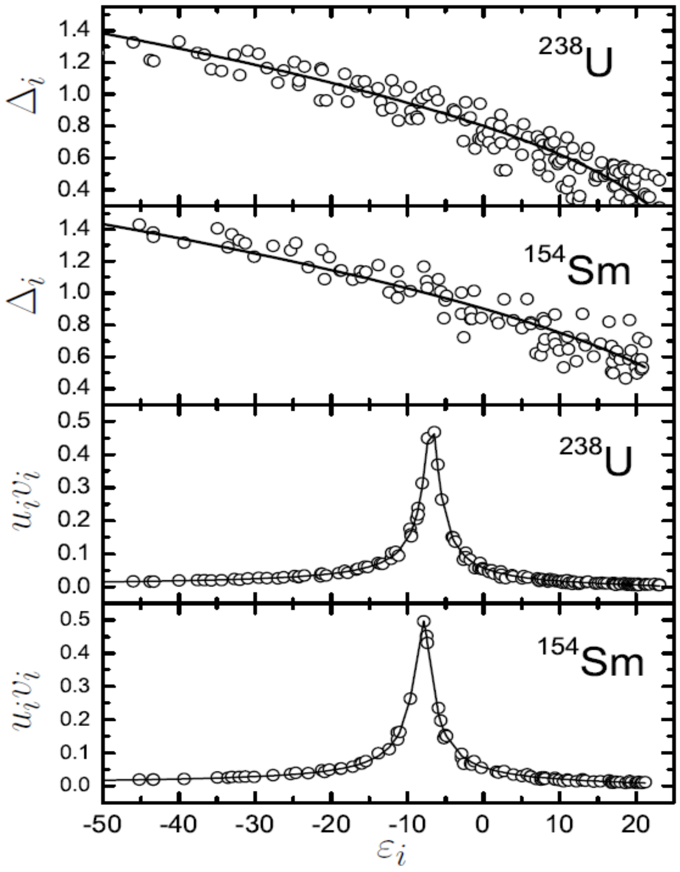
$$+ \; t_3(1+x_0 P_\sigma)\, \delta(\vec{r}_1-\vec{r}_2) \left[\rho\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)\right]^\alpha$$ Density dependent part

$$+ \; iW_{LS} \overleftarrow{\nabla}_{12} \delta(\vec{r}_1-\vec{r}_2) \times \overrightarrow{\nabla}_{12} \cdot (\vec{\sigma}_1+\vec{\sigma}_2)$$ Spin-orbit term

$$+ \; (1+2\tau_{1z})(1+2\tau_{2z}) \, \frac{e^2}{|\vec{r}_1-\vec{r}_2|}$$ Coulomb interaction between protons

Mapping of the Gogny force in the Canonical Basis

We fit the pairing strength *G* and the interaction cutoff α to the paring tensor $u_i v_i$ and the pairing gaps Δ_i of the Gogny HFB eigenstate in the Hartree-Fock basis.

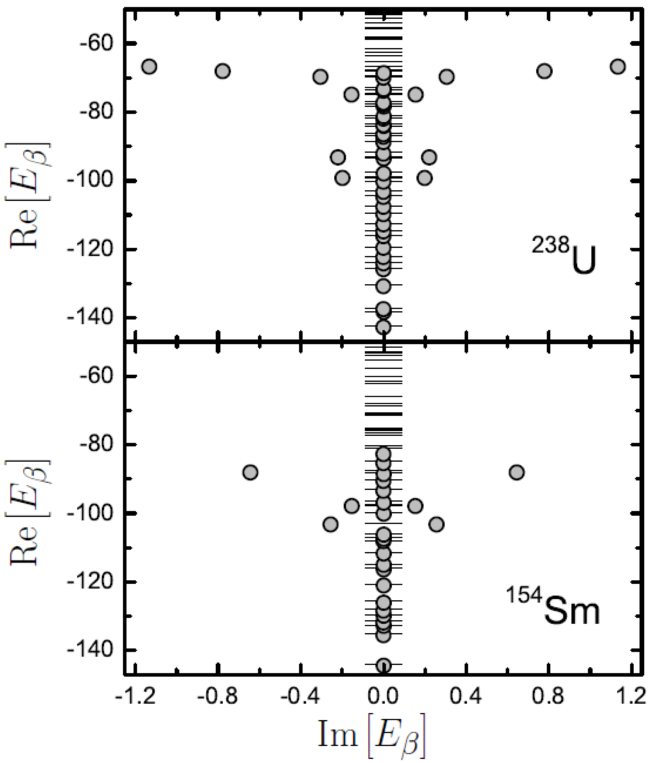


Protons

○ Gogny

– Hyperbolic

	M	L	D	G	α	Δ	E^{G}_{corr}	E^{BCS}_{corr}	E^{Exa}_{corr}
^{154}Sm	31	95	$9.9x10^{24}$	$2.2x10^{-3}$	32.7	0.158	1.3254	1.0164	2.9247
^{238}U	46	148	$4.8x10^{38}$	$2.0x10^{-3}$	25.3	0.159	0.861	0.503	2.651



Summary

- Two new realizations of the Hyperbolic model in condensed matter and nuclear structure.
- From the analysis of the exact Richardson wavefunction we proposed a new view to the nature of the Cooper pairs in the BCS-BEC transition for p-wave pairing.
- The hyperbolic RG offers a unique tool to study a rare 3º order QPT in the p_x+ip_y paired superfluid.
- We found that the root mean square size of the pair wave function diverges at the critical point. It could be a clear experimental signature of the QPT.
- The exactly solvable Hamiltonian with two free parameters reproduces the ground state properties of heavy nuclei as described by Gogny self-consistent mean field.
- It can be an excellent benchmark to test approximations beyond mean-field.
- Applications to other mesoscopic systems?