Thermalization under randomized local Hamiltonians

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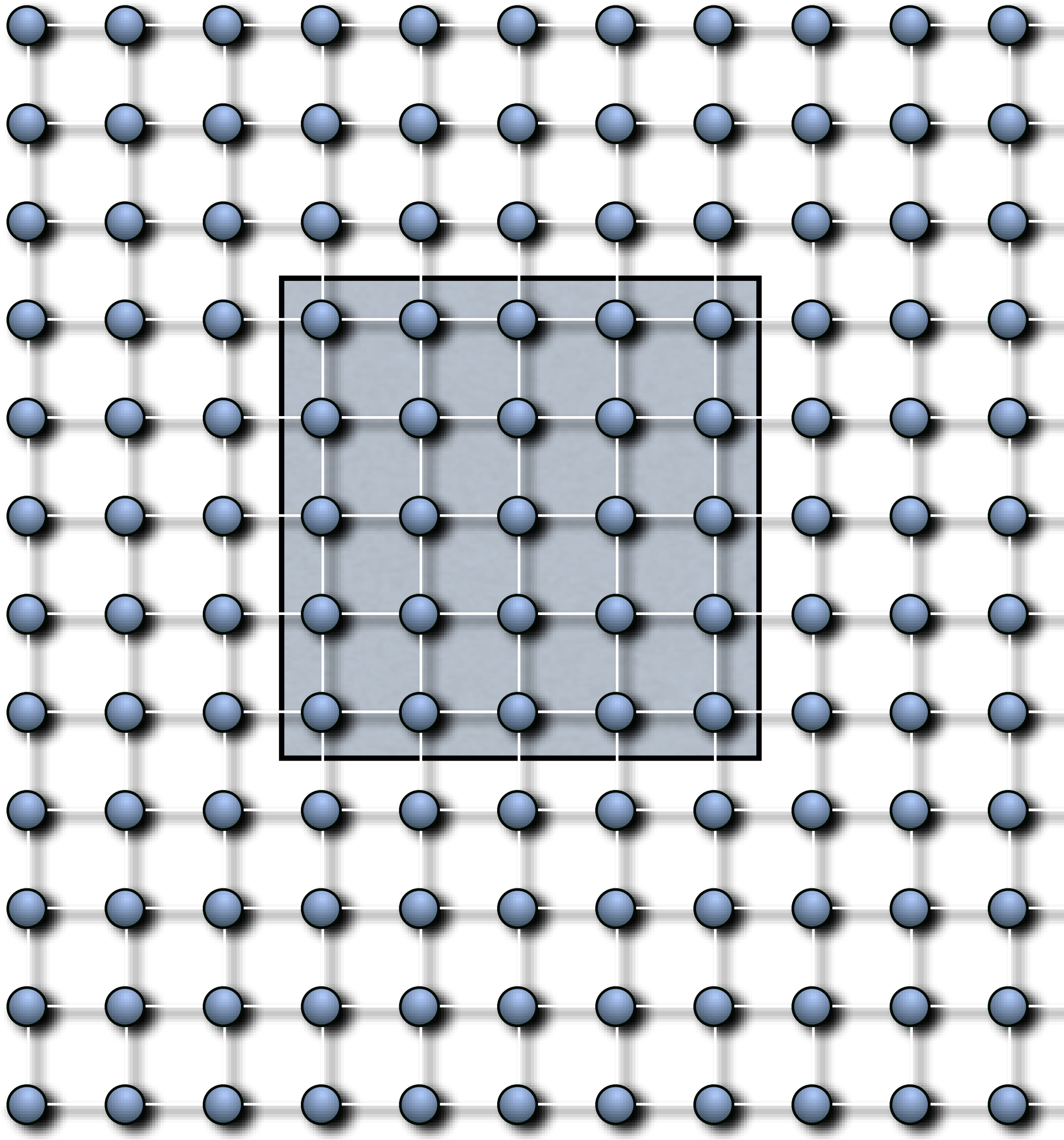
Why do systems thermalize?

lack of knowledge, ignorance → Jaynes' principle → $\hat{\varrho} = \mathrm{e}^{-\beta \hat{H}}/Z$

- Kinematic: $\hat{\varrho}_S = \mathrm{tr}_E \left[|\psi\rangle\langle\psi| \right]$
- Dynamic: $\hat{\varrho}_S(t) = \mathrm{tr}_E \left[|\psi(t)\rangle\langle\psi(t)| \right]$



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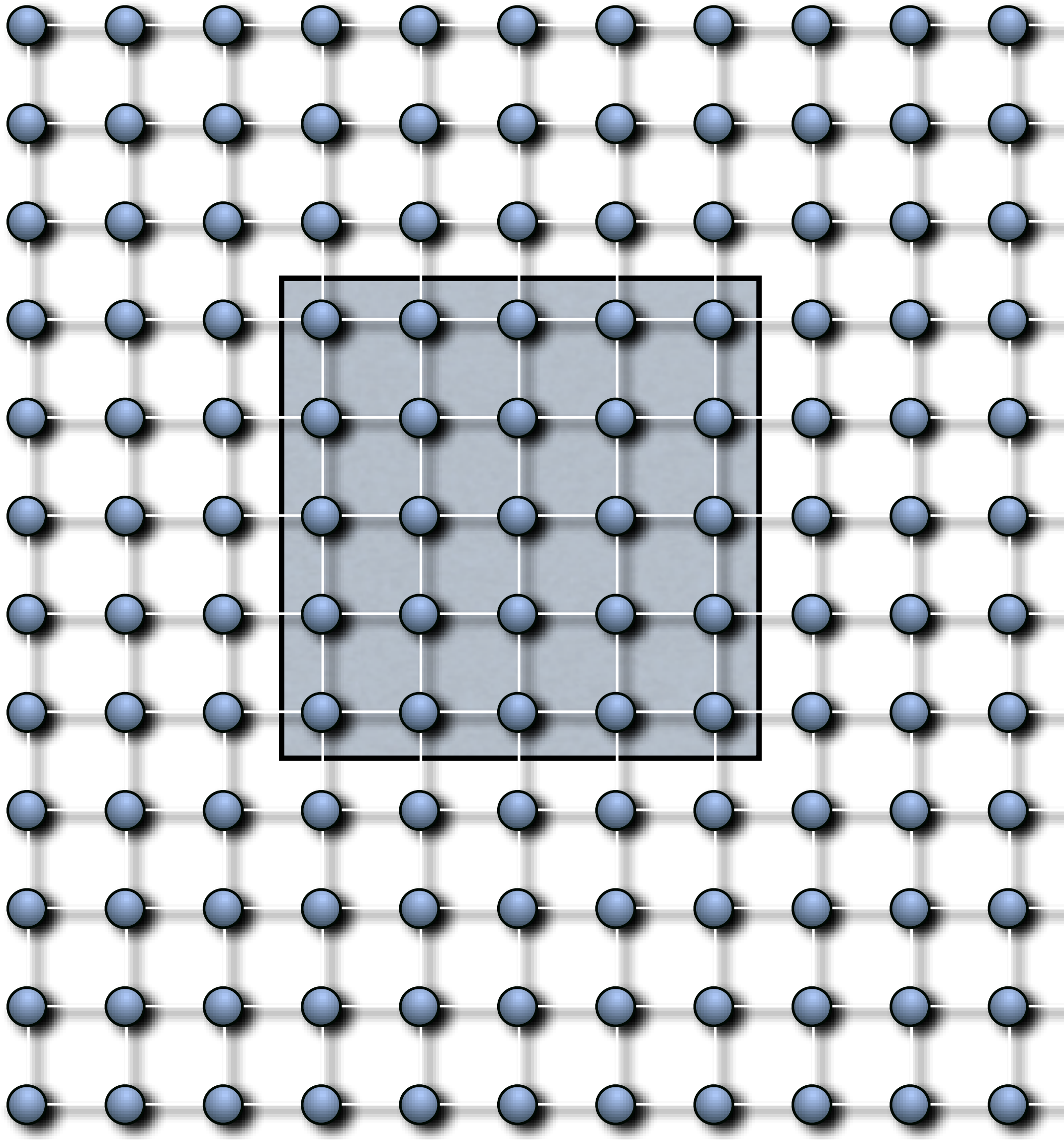
Popescu, Short, Winter, *Entanglement and the foundations of statistical mechanics*, Nat. Phys. **2**, 754 (2006).
Linden, Popescu, Short, Winter, *Quantum mechanical evolution towards thermal equilibrium*, Phys. Rev. E **79**, 061103 (2009).
Reimann, *Foundation of Statistical Mechanics under Experimentally Realistic Conditions*, Phys. Rev. Lett. **101**, 190403 (2008).

Dynamic: $\hat{\varrho}_S(t) = \mathrm{tr}_E\left[|\psi(t)\rangle\langle\psi(t)|\right]$, $|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\hat{H}t}|\psi_0\rangle$

$$\lim_{t\to\infty} \hat{\varrho}_S(t) = \textbf{?}$$



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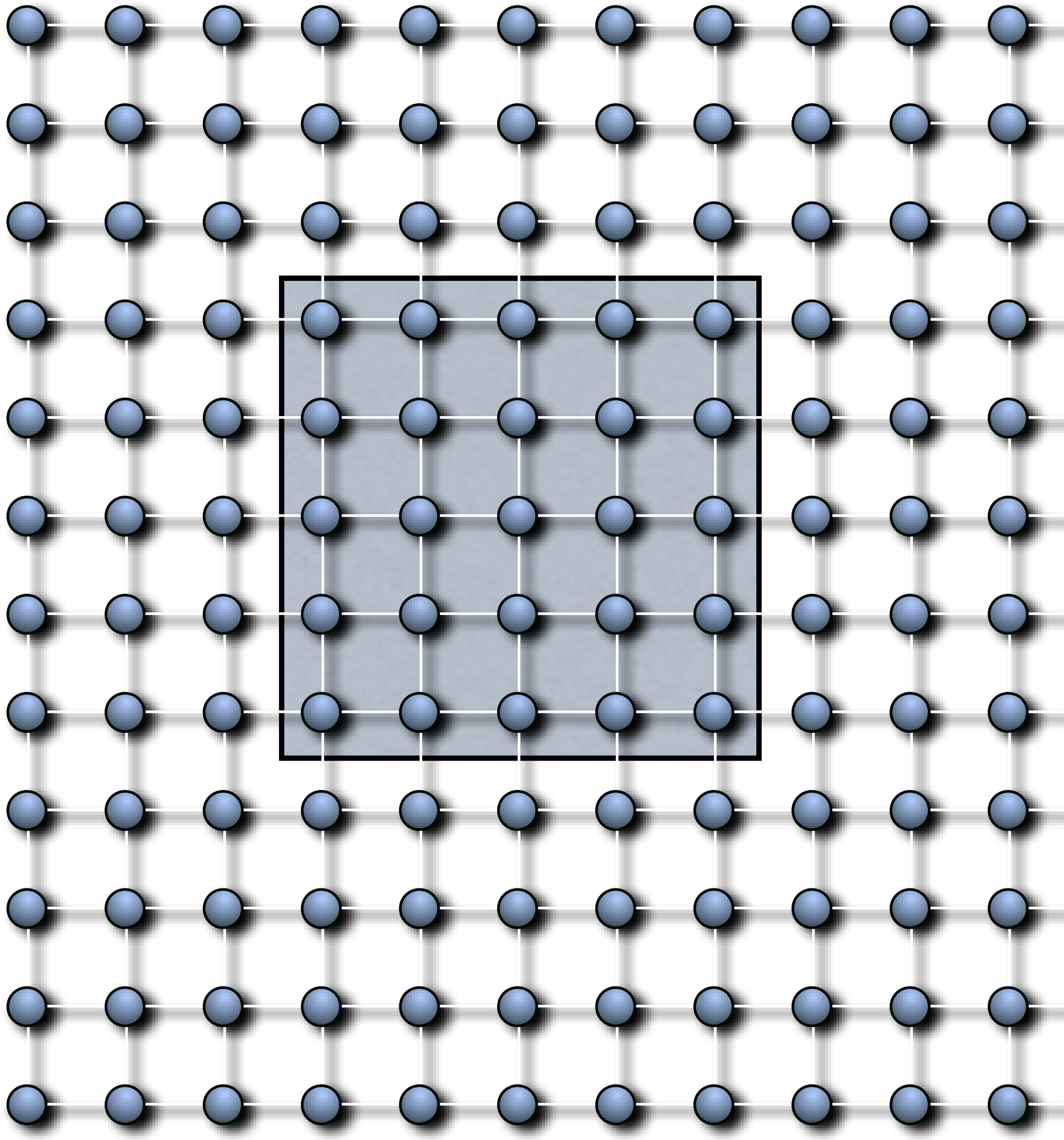
$\lim_{t\to\infty} \hat{\varrho}_S(t) =$ **?** if limit exists, it is equal to $\hat{\omega}_S = \mathrm{tr}_E[\hat{\omega}]$

$$\hat{\omega} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{d}t\, |\psi(t)\rangle\langle\psi(t)|$$



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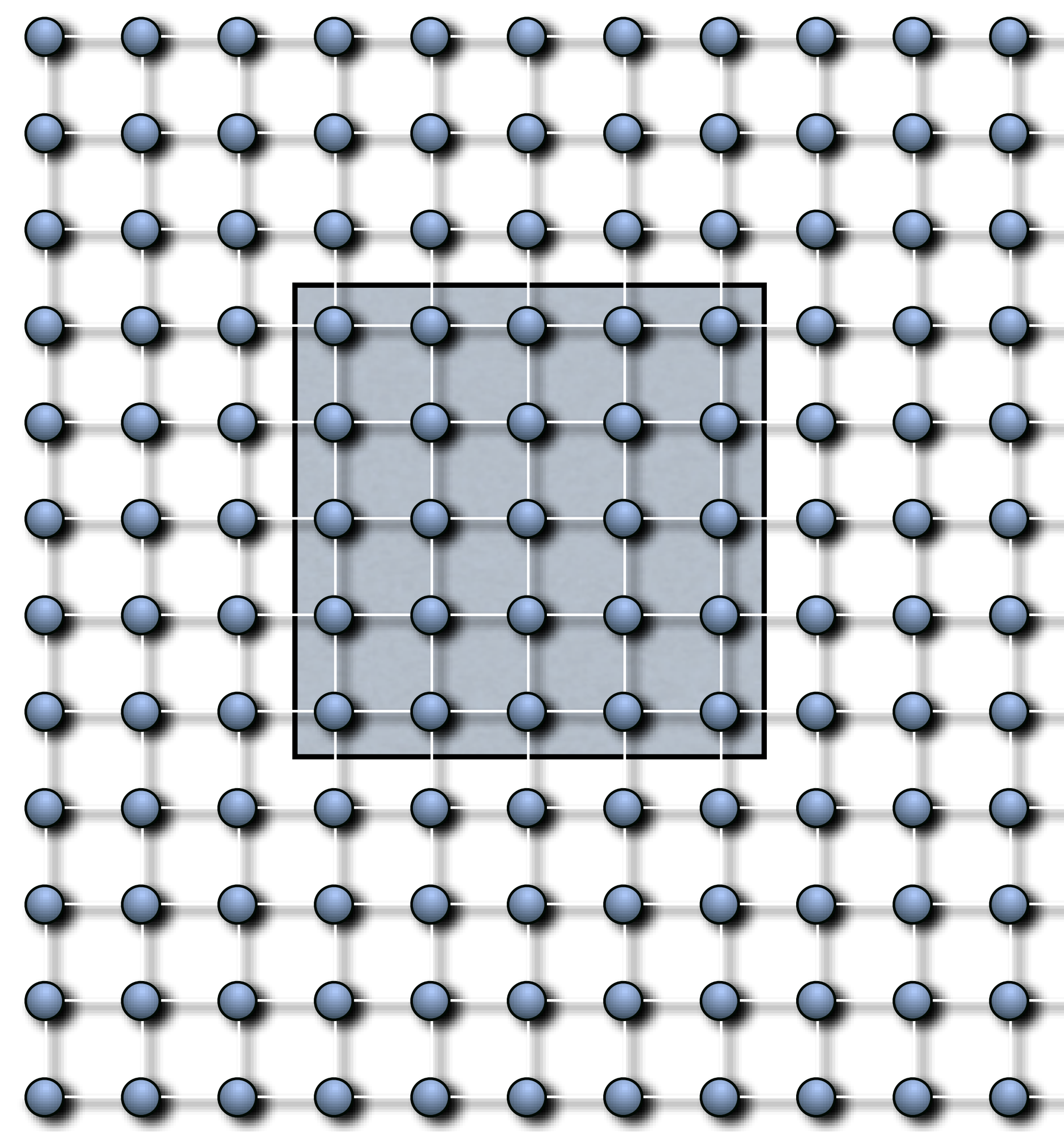
$$\delta(t) = \|\hat{\varrho}_S(t) - \hat{\omega}_S\|_{\mathrm{tr}}$$



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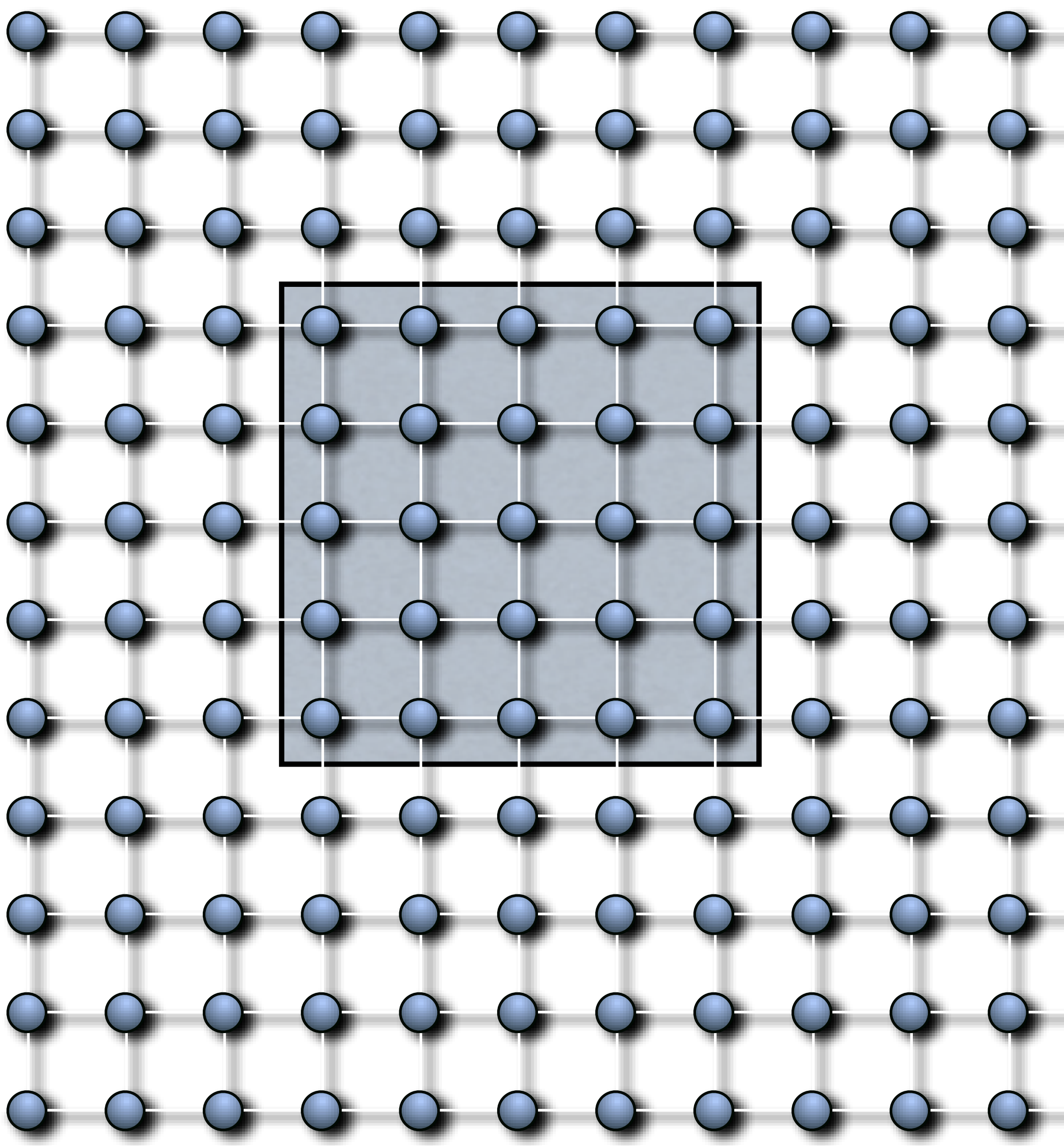

$$\delta(t) = \|\hat{\varrho}_S(t) - \hat{\omega}_S\|_{\mathrm{tr}}$$

$$\Delta(T) = \frac{1}{T}\int_0^T \mathrm{d}t\, \delta(t)$$

fraction of times in $[0,T]$ **for which** $\delta(t) \le \epsilon$ **is at least** $1 - \frac{\Delta(T)}{\epsilon}$

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Sudden quench: Equilibration

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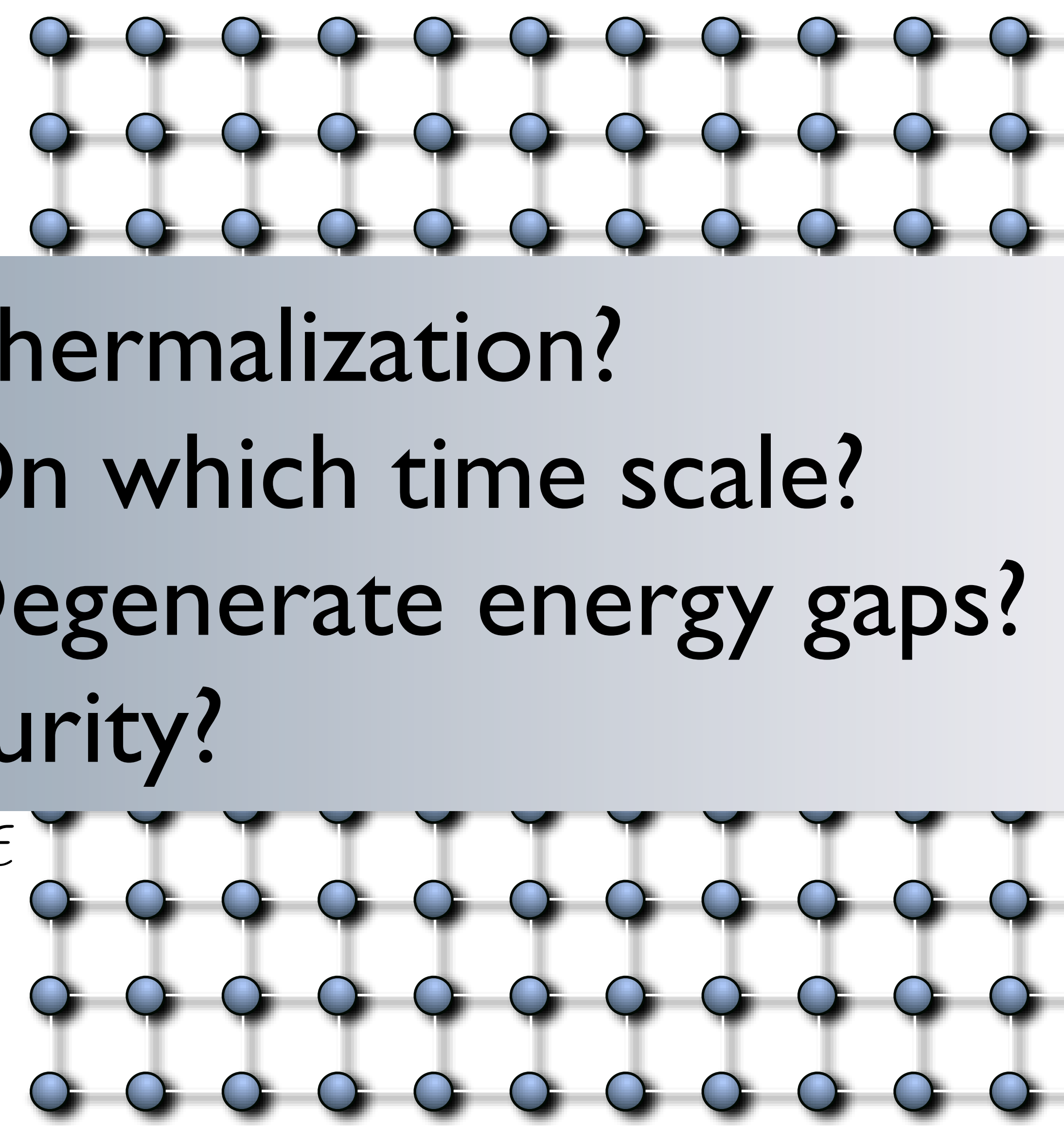
$$\hat{\omega} = \sum_n |\langle\psi_0|n\rangle|^2 |n\rangle\langle n|$$

non-degenerate energy gaps

$$\delta(t) = \|\hat{\varrho}_S(t) - \hat{\omega}_S\|_{\mathrm{tr}}$$

$$\Delta(\infty) \leq d_S\sqrt{\mathrm{tr}[\hat{\omega}^2]}$$

fraction of times for which $\delta(t) \leq \epsilon$ is at least $1 - \frac{d_S\sqrt{\mathrm{tr}[\hat{\omega}^2]}}{\epsilon}$



- Thermalization?
- On which time scale?
- Degenerate energy gaps?
- Purity?

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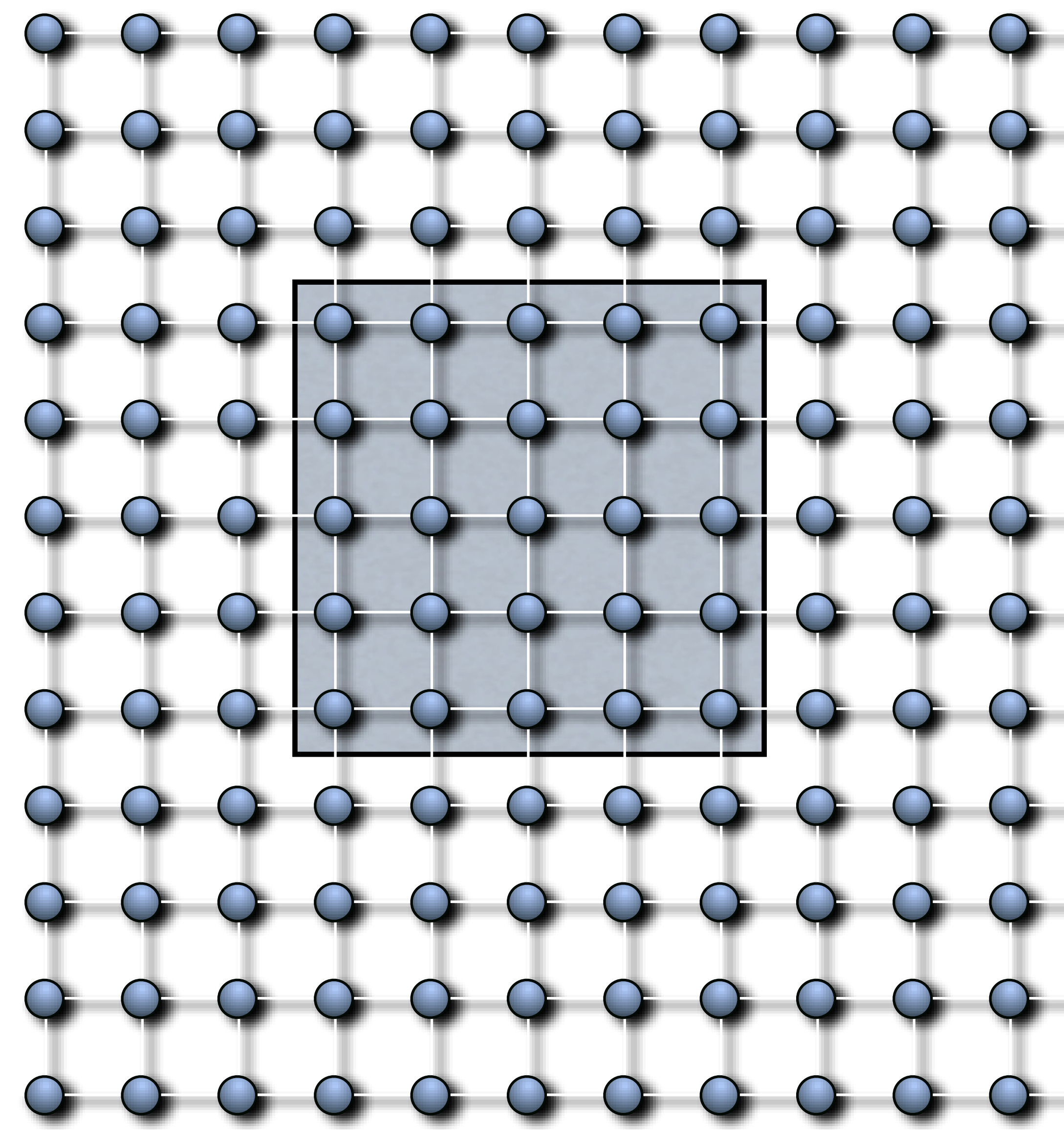

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Sudden quench to random Hamiltonians: Equilibration

$$\mathcal{H} = \mathrm{span}\{|\boldsymbol{n}\rangle = |n_1 \cdots n_N\rangle \,\big|\, n_i = 1, \ldots, d_i\} = \mathcal{H}_S \otimes \mathcal{H}_E$$

$$\hat{H} = \hat{U}\Big(\sum_{\boldsymbol{n}} E_{\boldsymbol{n}} |\boldsymbol{n}\rangle\langle\boldsymbol{n}|\Big)\hat{U}^\dagger \qquad \hat{\varrho}(t) = \mathrm{e}^{-\mathrm{i}t\hat{H}}\hat{\varrho}_0 \mathrm{e}^{\mathrm{i}t\hat{H}}$$

Brandão, Ćwikliński, Horodecki, Horodecki, Korbicz, Mozrzymas, *Convergence to equilibrium under a random Hamiltonian*, arXiv:1108.2985.
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fix $E_{\boldsymbol{n}}$, pick $\hat{U}$ at random according to the Haar measure, denote $\mathbb{E}[\bullet] = \int \bullet \, \mathrm{d}\mu(\hat{U})$ and recall $\delta(t) = \|\hat{\varrho}_S(t) - \hat{\omega}_S\|_{\mathrm{tr}}$

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$$\mathbb{E}[\delta(t)] \leq \sqrt{d_S}\sqrt{|\phi(t)|^4 + 4/d_E}$$

- FT of spectral density $\phi(t) = \mathrm{tr}[\frac{\mathbb{1}}{d}\mathrm{e}^{\mathrm{i}t\hat{H}}]$
- Loschmidt echo $\mathrm{tr}[|\psi\rangle\langle\psi|\mathrm{e}^{\mathrm{i}t\hat{H}}]$
- Characteristic function $\mathrm{tr}[\hat{\varrho}\mathrm{e}^{\mathrm{i}t\hat{H}}]$

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explicit expression for expected purity of $\hat{\varrho}_S(t)$

Vinayak, Znidaric, *Subsystem dynamics under random Hamiltonian evolution*, arXiv:1107.6035 fix E_n, pick $\hat{U}$ at random according to the Haar measure, denote $\mathbb{E}[\bullet] = \int \bullet \, \mathrm{d}\mu(\hat{U})$ and recall $\delta(t) = \|\hat{\varrho}_S(t) - \frac{\mathbb{1}_S}{d_S}\|_{\mathrm{tr}}$,

$\Delta(T) = \frac{1}{T}\int_0^T \mathrm{d}t \, \delta(t)$

$\mathbb{E}\left[\Delta(T)\right] \leq c$ implies that with probability at least $1-\epsilon'$ the fraction of times in $[0,T]$ for which $\delta(t) \leq \epsilon$ is at least $1 - \frac{c}{\epsilon\epsilon'}$

↓ $\frac{c}{\epsilon\epsilon'} \ll 1$

for almost all unitaries and almost all times in $[0,T]$ the system is close to the maximally mixed state

$$\mathbb{E}[\delta(t)] \leq \sqrt{d_S}\sqrt{|\phi(t)|^4 + 4/d_E}$$

solvable system: $E_{\boldsymbol{n}} = \sum_{k=1}^{N} \epsilon_k n_k, \quad n_k = 0, 1$

e.g.: $\hat{H} = \sum_{i=1}^{N-1} \sum_{\alpha,\beta=x,y} J_{i,\alpha,\beta}\, \hat{\sigma}_i^{\alpha} \hat{\sigma}_{i+1}^{\beta} - \sum_{i=1}^{N} h_i \hat{\sigma}_i^z$

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for $|t| \leq c_0$ **one has** $|\phi(t)| \leq \mathrm{e}^{-\sigma^2 t^2/2}$

$$\sigma^2 = \left\langle \left(\hat{H} - \langle \hat{H} \rangle\right)^2 \right\rangle / N$$

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for $T \leq c_0$ one has $\mathbb{E}\big[\Delta(T)\big]^2 \leq a_0 d_S \left(\frac{1}{\sqrt{N} T \sigma} + \frac{1}{d_E}\right)$

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e.g.: $\hat{H} = \sum^{N-1} \sum J \, \hat{\sigma}^{\alpha} \hat{\sigma}^{\beta} - \sum^{N} h \, \hat{\sigma}^{z}$

for sufficiently large system size, quenches under almost all Hamiltonians with $E_{\boldsymbol{n}} = \sum_k \epsilon_k n_k$ lead to thermalization (the subsystem spends most of its time in $[0, T]$ close to the maximally mixed state) in a time $T \propto N^{\epsilon - 1/2}$

in particular $\mathbb{E}\left[\Delta(T)\right]^2 \leq a_0 d_S \left(\frac{1}{N^{\epsilon}\sigma} + \frac{1}{d_E}\right)$

for $T \propto N^{\epsilon-1/2}$ and all $0 < \epsilon \leq 1/2$

$$\sigma^2 = \left\langle \left(\hat{H} - \langle \hat{H} \rangle\right)^2 \right\rangle / N$$

$$\mathbb{E}[\delta(t)] \leq \sqrt{d_S}\sqrt{|\phi(t)|^4 + 4/d_E}$$

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quantum central limit theorems: $\lim_{N\to\infty} \mathrm{tr}[\hat{\varrho}\mathrm{e}^{\mathrm{i}s\hat{A}}] = \mathrm{e}^{-\sigma^2 s^2/2}$

Goderis, Vets, *Central limit theorem for mixing quantum systems and the CCR-algebra of fluctuations*, Commun. Math. Phys. **122**, 249 (1989).
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Berry-Esseen theorem: $|F(x) - G(x)| \leq \frac{c}{\sqrt{N}}$

quantum version:

- Observables have Gaussian distribution
- Bound on purity (effective dimension) $\mathrm{tr}[\hat{\omega}^2]$

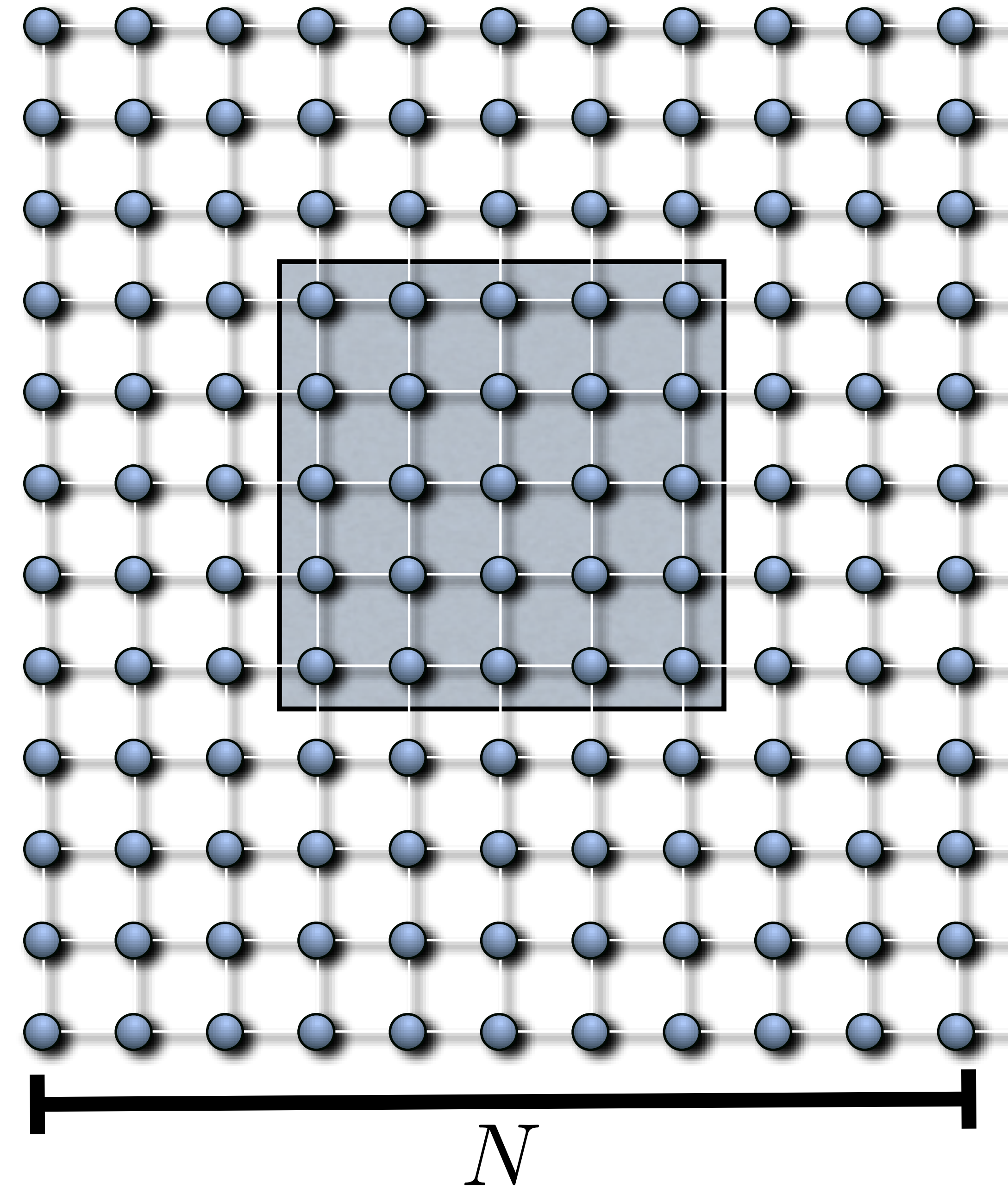
M. Cramer, M. Guta, in preparation

$$\mathbb{E}[\delta(t)] \leq \sqrt{d_S}\sqrt{|\phi(t)|^4 + 4/d_E}$$

E_n spectrum of local Hamiltonian on D-dimensional cubic lattice:

$$\mathbb{E}\left[\Delta(T)\right]^2 \leq a_0 d_S \left(\frac{1+1/\sigma^3}{N^{1/(5D)}} + \frac{1}{d_E}\right)$$

for $T \propto \sigma^2 N^{1/(5D)-1/2}$



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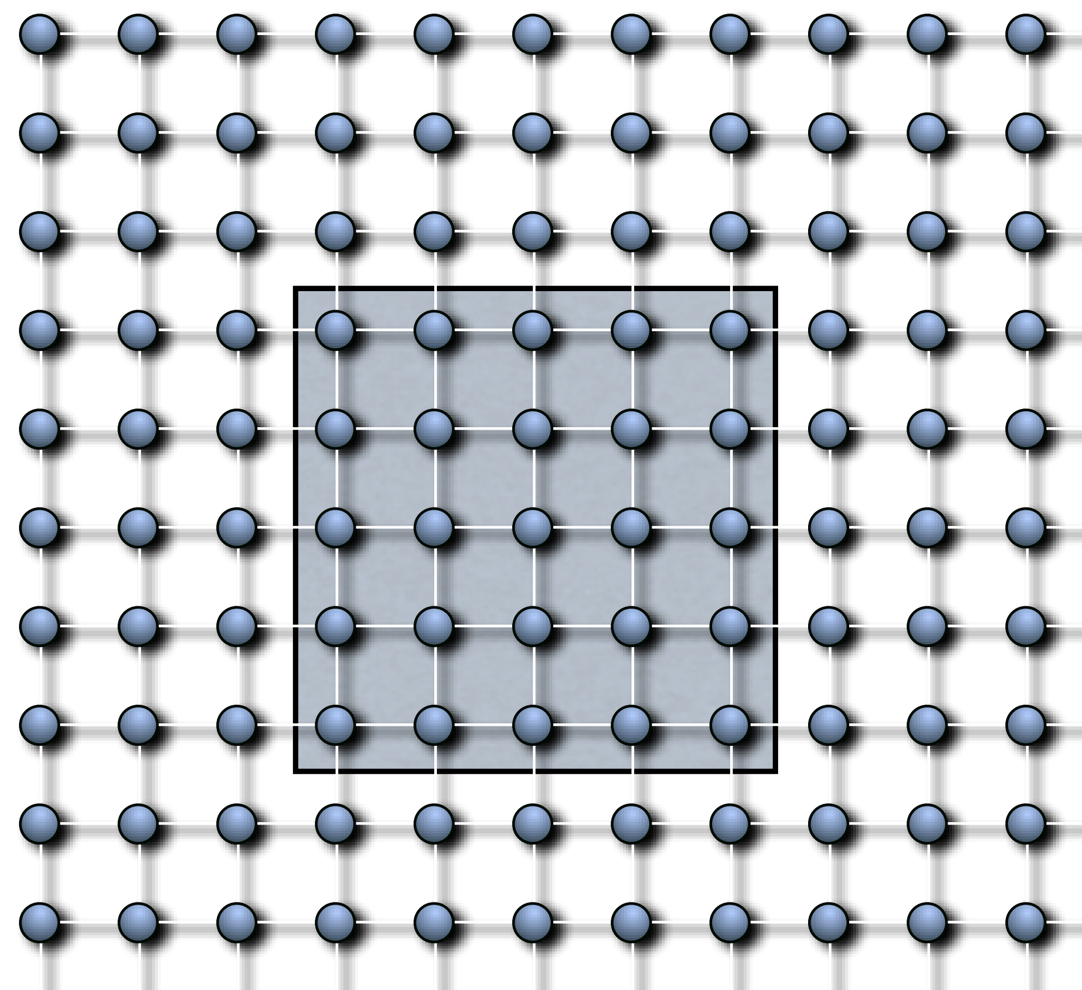

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for sufficiently large system size, quenches under almost all Hamiltonians that are unitarily equivalent to a local Hamiltonian lead to thermalization (the subsystem spends most of its time in $[0,T]$ close to the maximally mixed state) in a time $T \propto N^{1/(5D)-1/2}$

subsystem may even be of size $\propto \log(N)$