Exactly solvable models in the cold atomic systems

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Content

I. Exactly solvable models with anisotropic spin-exchanging interaction

Spin-1/2 bose gas

Spin-1 bose gas

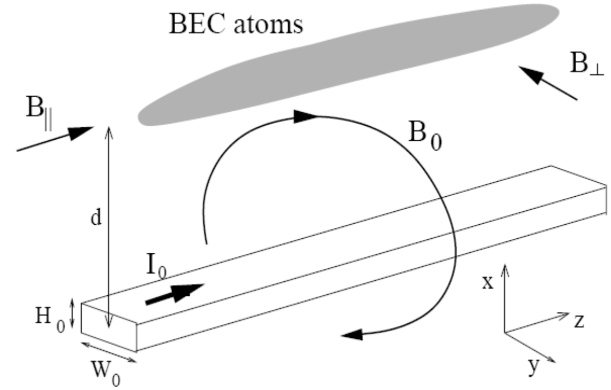
Spin-3/2 fermi gas

II. Exactly solvable models with spin-orbital coupling

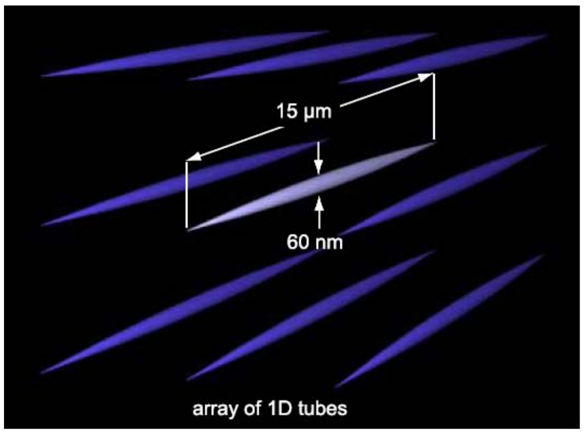
I. Exactly solvable models with anisotropic spin-exchanging interaction

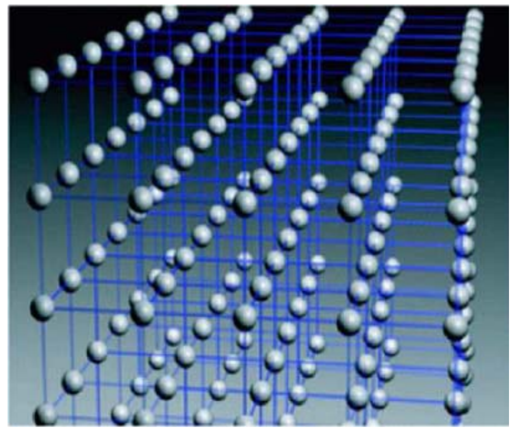
low-dimensional cold atomic systems

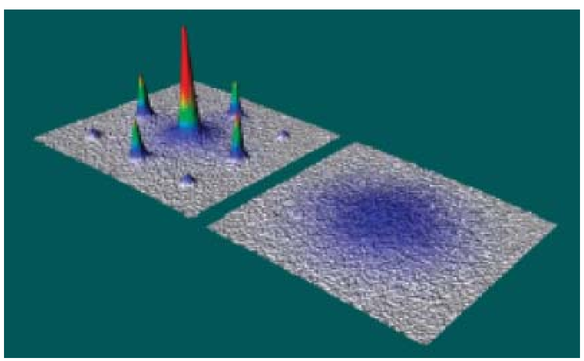
Magnetic traps
& Feshbach resonances

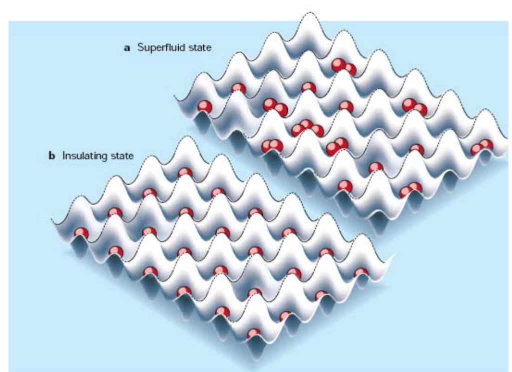


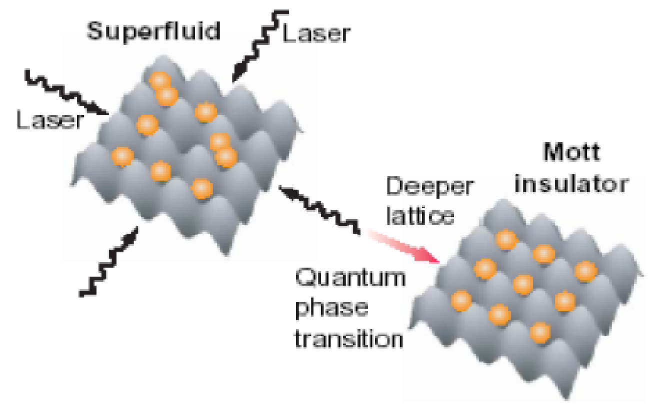
Optical lattices











Tunable: component, interaction, dimension, lattice constant

Exactly solvable cold atomic models

- Hamiltonian $H=-\sum_{j=1}^{N}\partial_{x_j}^2+\sum_{\langle i,j\rangle}\sum_{lm} g_{lm} P_{ij}^{lm}\delta(x_i-x_j).$

	(pseudo-)spin	interaction	symmetry	
★	0 (boson)	$g_0=c$	$U(1)$	Lieb, *et al.*, PR130.1605(1963)
★	½ (fermion)	$g_0=c$	$SU(2)$	Yang, PRL19.1312(1967)
★	½ (boson)	$g_1=c$	$SU(2)$	Li, EPL 61. 368 (2003)
	½ (boson)	$g_{1,-1}=c_1, g_{1,1}=c_2, g_{1,0}=0.$	$U(1)$	
★	1 (boson)	$g_0=c, g_2=c$	$SU(3)$	Zhou, JPA 21. 2391; 2399 (1988)
■	1 (boson)	$g_0=-c, g_2=2c.$	$SU(2)$	Cao, *et.al*, EPL79.30005(2007)
	1 (boson)	$g_{0,0}=c, g_{2,-1}=0, g_{2,1}=0,$ $g_{2,-2}=c, g_{2,0}=c, g_{2,2}=c.$	$U(1)$	
★	1 (fermion)	$g_1=c$	$SU(3)$	Sutherland, PRL20.98(1968)
★	3/2 (fermion)	$g_0=c, g_2=c.$	$SU(4)$	Sutherland, PRL20.98(1968)
■	3/2 (fermion)	$g_0=3c, g_2=c.$	$Sp(4)$	Jiang, *et.al*, EPL87.10006(2009)
	3/2 (fermion)	$g_{0,0}=0, g_{2,2}=c_1, g_{2,-2}=c_2,$ $g_{2,-1}=0, g_{2,0}=0, g_{2,-1}=0,$	$U(1)$	
■	Integer s (boson)	$g_0=-(s-1/2)c, g_{2,4,\ldots}=c.$	$SO(2s+1)$	Jiang, *et.al*, JPA44.345001(2011)
■	Half-odd s (fermion)	$g_0=(s+3/2)c, g_{2,4,\ldots}=c.$	$Sp(2s+1)$	Jiang, *et.al*, JPA44.345001(2011)

1. Anisotropic spin-1/2 bose gas

Anisotropic spin-exchanging interaction

1). Spin-½ fermions

$$\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}^{N} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \Delta \sigma_i^z \sigma_j^z) \delta(x_i - x_j)$$

Contact interaction: non-integrable;

Heisenberg & long range interactions, i.e. $1/r$ & $1/r^2$: integrable.

Spin-1/2 bosons : non-integrable.

2). Cold atoms : spin-½ bosons

$$\begin{aligned} \hat{H} &= -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j}^{N} (c_1 \hat{P}_{ij}^{1,1} + c_2 \hat{P}_{ij}^{1,-1}) \delta(x_i - x_j) + \sum_{j=1}^{N} h \hat{\sigma}_j^z \\ &= -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{1}{4} \sum_{i \neq j}^{N} \left[(c_1 + c_2)(1 + \sigma_i^z \sigma_j^z) + (c_1 - c_2)(\sigma_i^z + \sigma_j^z) \right] \delta(x_i - x_j) - \sum_{j=1}^{N} h \hat{\sigma}_j^z \end{aligned}$$

Exact solutions

$$\hat{S}_{ab}(k) = \frac{k - \mathrm{i}c_1}{k + \mathrm{i}c_1}\hat{P}_{ab}^{1,1} + \frac{k - \mathrm{i}c_2}{k + \mathrm{i}c_2}\hat{P}_{ab}^{1,-1} + P_{ab}^{0,0} + P_{ab}^{1,0}$$

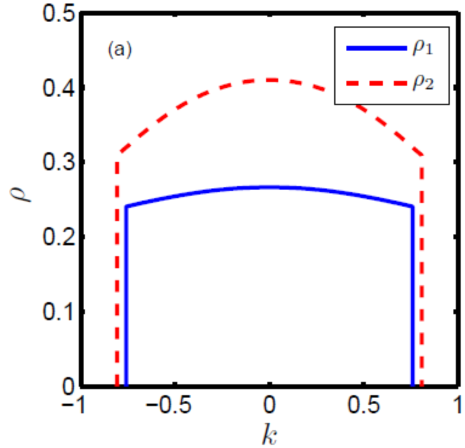
$$E = \sum_{i=1}^{2}\sum_{j=1}^{N_i} {k_j^{(i)}}^2 - hM^z, \;\; K = \sum_{i=1}^{2}\sum_{j=1}^{N_i} k_j^{(i)}$$

$$\mathrm{e}^{\mathrm{i}k_j^{(i)}L} = \prod_{l \neq j}^{N_i} \frac{k_j^{(i)} - k_l^{(i)} + \mathrm{i}c_i}{k_j^{(i)} - k_l^{(i)} - \mathrm{i}c_i},$$

$$j = 1, 2, \cdots, N - M, \;\; i = 1, 2.$$

$$M^z = (N_1 - N_2)/2$$

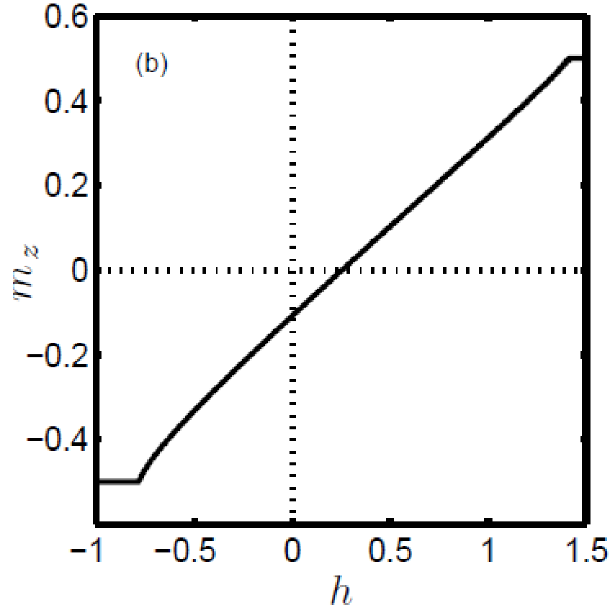
Densities distribution of quasi-momentum



$c_1 = 1$, $c_2 = 0.5$, $n = 1$ and $h = 0$.

Magnetization $m_z = (n_1 - n_2)/2.$

interaction



Spontaneous magnetization when h=0

Phase transition from fully polarized state to partially polarized state.

Critical points with strong repulsion

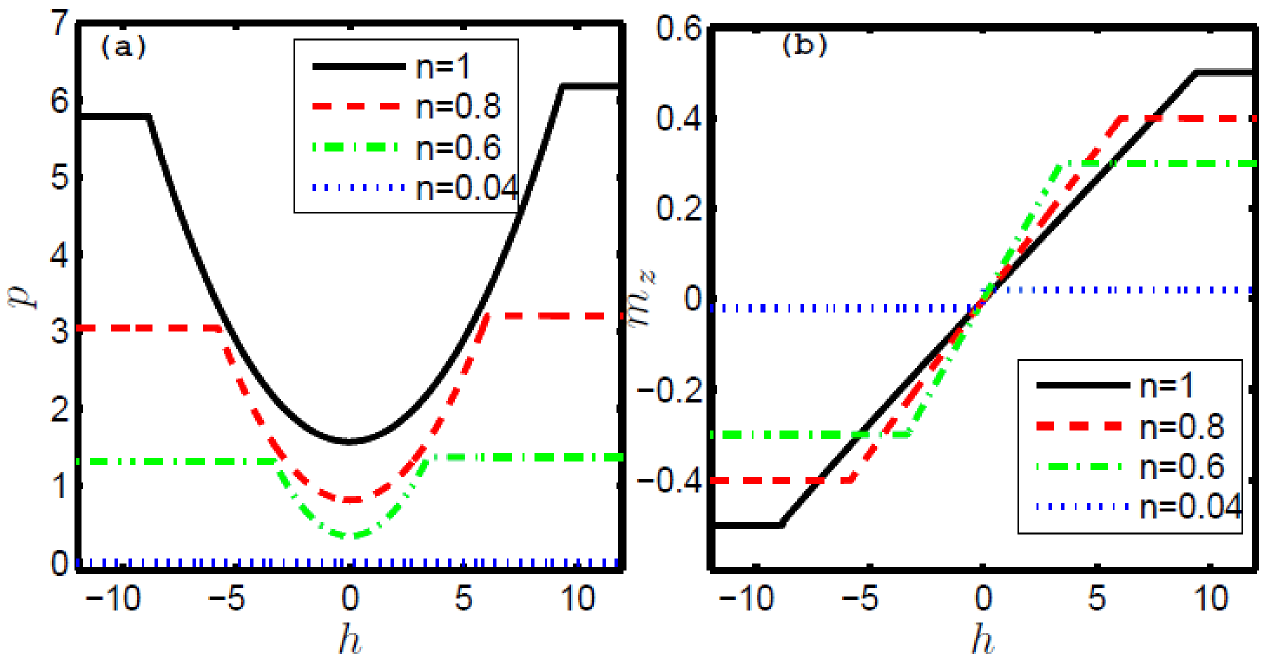
$$h_{c+} = \pi^2 n^2 - 8\pi^2 n^3/3c_1,$$
$$h_{c-} = \pi^2 n^2 - 8\pi^2 n^3/3c_2.$$

The critical points are different because the couplings c1 and c2 are different.

The pressures & magnetization in the strong coupling limit

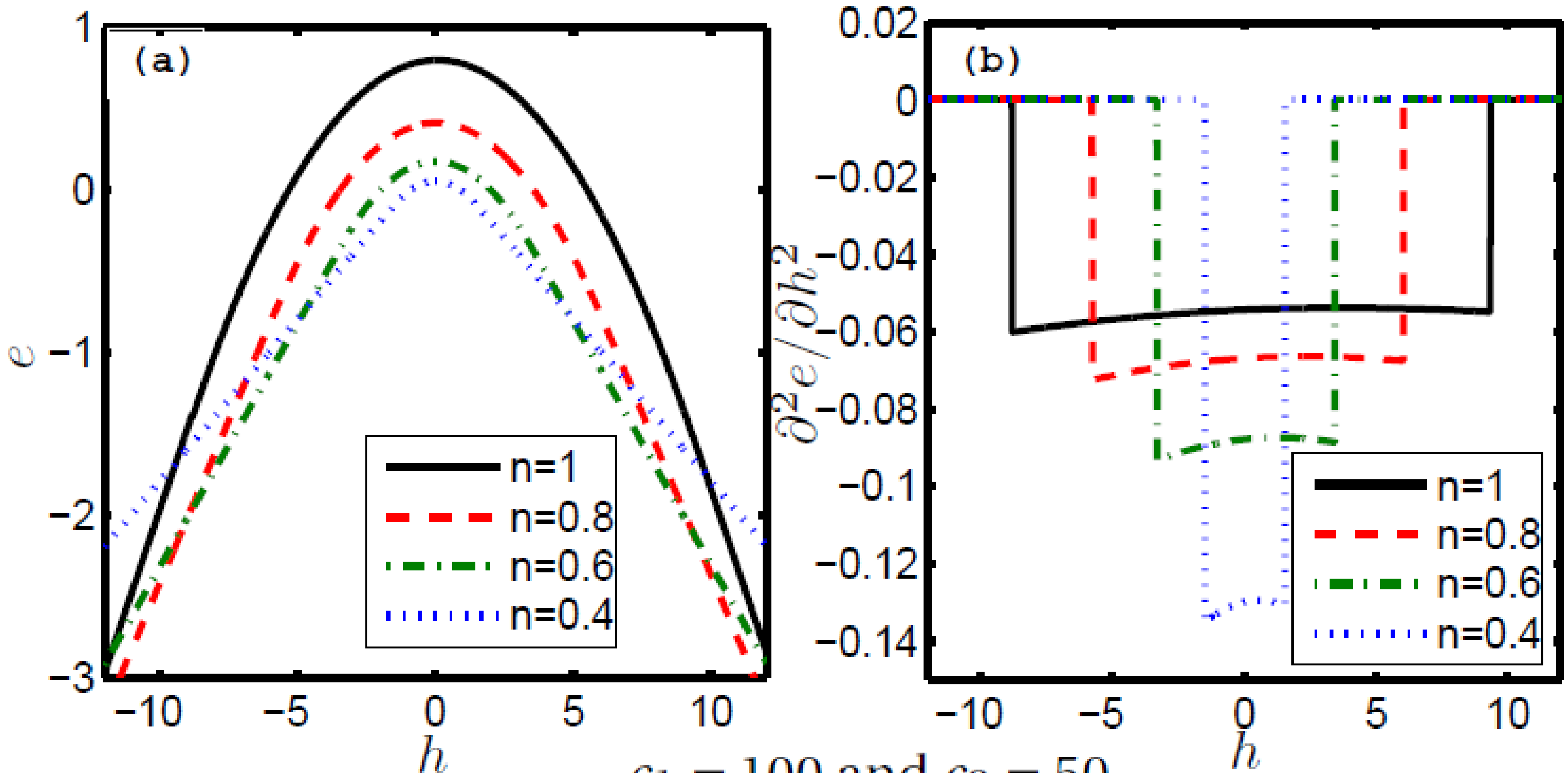
$$p = \frac{1}{6}\pi^2 n^3 + \frac{h^2}{2n^2\pi^2} - \frac{\pi^2}{12c_1}\left(n + \frac{h}{n\pi^2}\right)\left(3n - 5\frac{h}{n\pi^2}\right) - \frac{\pi^2}{12c_2}\left(n - \frac{h}{n\pi^2}\right)\left(3n + 5\frac{h}{n\pi^2}\right) + O(1/c_1^2) + O(1/c_2^2),$$

$$m_z = \frac{h}{2n^2\pi^2} + \frac{1}{3n^2c_1}\left(n + \frac{h}{n\pi^2}\right)^3 - \frac{1}{3n^2c_2}\left(n - \frac{h}{n\pi^2}\right)^3 + O(1/c_1^2) + O(1/c_2^2)$$



When the external field h is zero, the pressure takes its minimum. With the increasing h, the pressure increases. At the fully polarized state, the pressure arrives at its maximum.

The ground state energy density

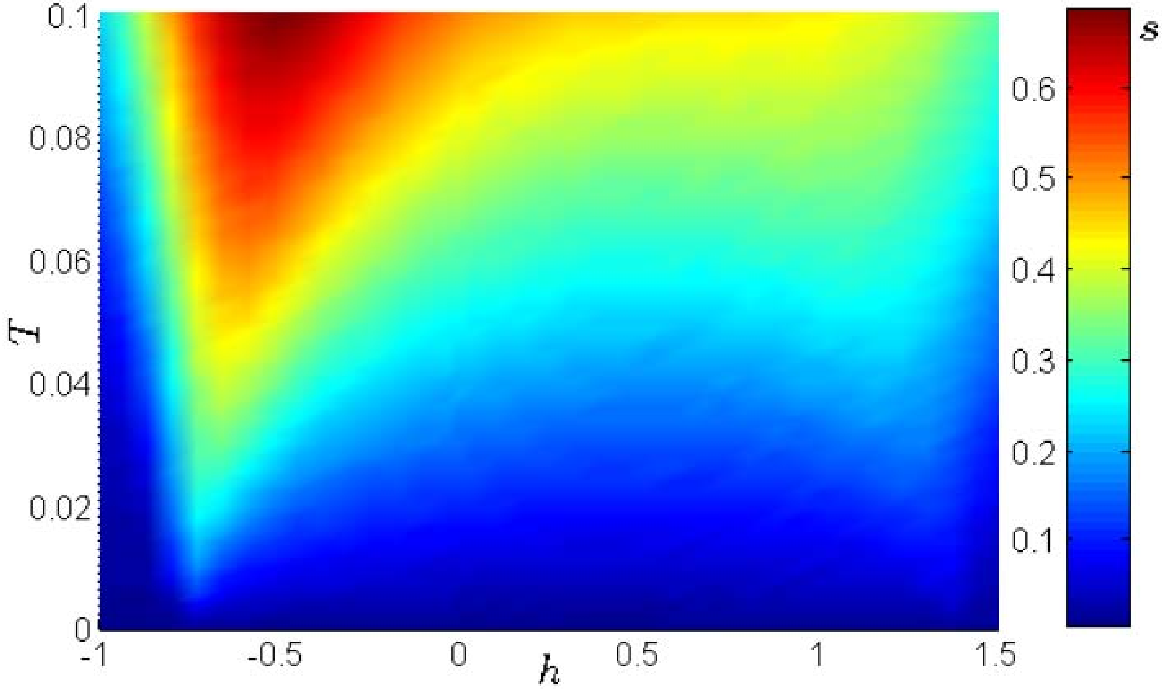


$c_1 = 100$ and $c_2 = 50$

$$E = \sum_{i=1}^{2} \sum_{j=1}^{N_i} {k_j^{(i)}}^2 - hM^z$$

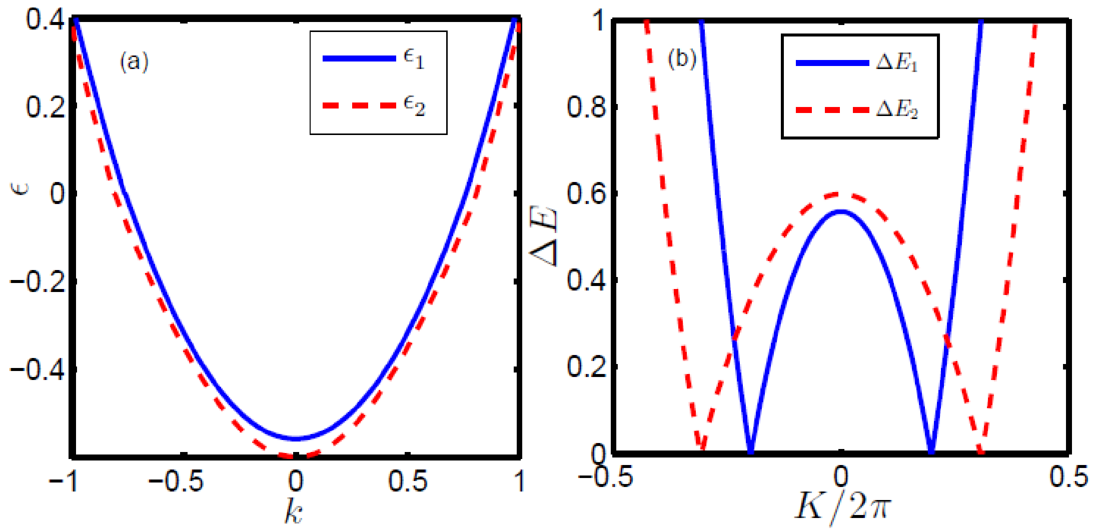
At the critical point, the second order derivative of the ground state energy density is not continue, thus it is a second order phase transition.

Entropy at finite temperature



$c_1 = 1$, $c_2 = 0.5$ and $n = 1$

Dressed energy $$\varepsilon_i(k) = T\ln\left[\eta_i^{\mathrm{h}}(k)/\eta_i(k)\right]$$



$c_1 = 1$

$c_2 = 0.5$

$n = 1$

$h = 0$

Strong repulsion

$$\varepsilon_i(k) = k^2 - \mu_i - c_i^{-1}T^{3/2}\pi^{-1/2}\mathrm{F}_{1/2}(\mu_i/T) + O(c_i^{-2})$$

Fermi-Dirac function

$$\mathrm{F}_j(x) = \Gamma^{-1}(j+1)\int_0^\infty y^j/(\mathrm{e}^{y-x}+1)\mathrm{d}y$$

Scaling behavior near the critical point

Grand canonical ensemble

$$\mu_{c+} = -h/2, \quad \mu_{c-} = h/2,$$

Near critical point, density of particle number satisfies scaling law

$$n(\mu, h, T) - n_0(\mu, h, T) = T^{d/z+1-1/\nu z} \mathscr{F}\left(\frac{\mu - \mu_{c+}}{T}\right)$$

$$d/z + 1 - 1/\nu z = 1/2, \ 1/\nu z = 1,$$

$$\mathscr{F}(x) = \mathrm{F}_{-1/2}[(\mu - \mu_{c+})/T]/(2\pi^{1/2})$$

Focus on this model,

dimension *d*=1,
critical exponent *z*= 2,
correlation length exponent *ν*= 1/2

2. Anisotropic spin-1 Bose gas

Hamiltonian

$$\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + c\sum_{i\neq j}^{N} (\hat{P}_{ij}^{0,0} + \hat{P}_{ij}^{2,2} + P_{ij}^{2,0} + P_{ij}^{2,-2})\delta(x_i - x_j)$$

$$= -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \frac{c}{2}\sum_{i\neq j}^{N} [(\hat{s}_i^x \hat{s}_j^x + \hat{s}_i^y \hat{s}_j^y)^2 + 3(\hat{s}_i^z \hat{s}_j^z)^2 - (\hat{s}_i^z)^2 - (\hat{s}_j^z)^2 + \hat{s}_i^z \hat{s}_j^z]\delta(x_i - x_j).$$

Scattering matrix

$$\hat{S}_{ab}(k) = \frac{k - ic}{k + ic}(\hat{P}_{ij}^{0,0} + \hat{P}_{ij}^{2,2} + P_{ij}^{2,0} + P_{ij}^{2,-2}) + P_{ab}^{1} + P_{ab}^{2,1} + P_{ab}^{2,-1}$$

Energy spectrum

$$E = \sum_{j=1}^{M^1} {k_j^{(1)}}^2 + \sum_{j=1}^{M^2} {k_j^{(2)}}^2,$$

$$S = M^2 - 2M^3.$$

Bethe ansatz equations

$$e^{ik_j^{(1)}L} = \prod_{l\neq j}^{M^1} \frac{k_j^{(1)} - k_l^{(1)} + ic}{k_j^{(1)} - k_l^{(1)} - ic}, \quad j = 1, 2, \cdots, M^1,$$

$$e^{ik_j^{(2)}L} = \prod_{l\neq j}^{M^2} \frac{k_j^{(2)} - k_l^{(2)} + ic}{k_j^{(2)} - k_l^{(2)} - ic} \prod_{l}^{M^3} \frac{k_j^{(2)} - \lambda_l - ic/2}{k_j^{(2)} - \lambda_l + ic/2},$$

$$j = 1, 2, \cdots, M^2,$$

$$\prod_{l=1}^{M^2} \frac{\lambda_j - k_l^{(2)} - ic/2}{\lambda_j - k_l^{(2)} + ic/2} = \prod_{l\neq j}^{M^3} \frac{\lambda_j - \lambda_l - ic}{\lambda_j - \lambda_l + ic}, \quad j = 1, 2, \cdots, M^3.$$

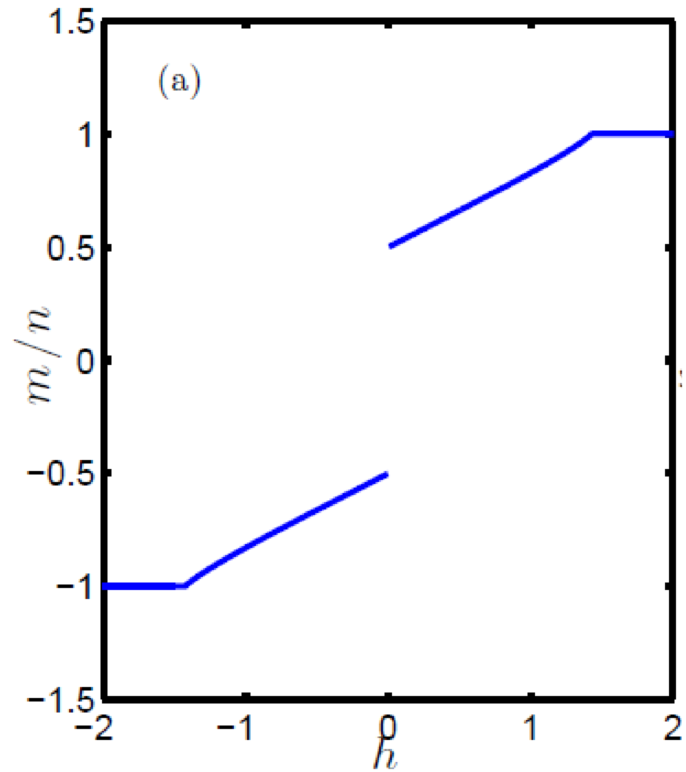
Solutions of the Bethe sansatz equations with the repulsive interaction

$$k_z^{(1)}, \quad k_z^{(1)} \in \mathbb{R}; \qquad k_z^{(2)}, \quad k_z^{(2)} \in \mathbb{R};$$

$$\lambda_{2z}^{(2)} = k_{2z}^{(2)}, \quad k_{2z}^{(2)} \in \mathbb{R};$$

$$\lambda_{nzj} = \lambda_{nz} + (n + 1 - 2j)ic/2, \ j = 1, 2, \cdots, n, \ n = 1, 2, \cdots$$

Ground state



$n = 1$ and $c = 1$

$$h = 0, \qquad |m| = n/2.$$

half of the atoms occupy the state with $s^z=0$ and the rest stay on the state with $s^z=1$ (or -1).

fully polarized state

$$h > |h_c|, \qquad S^z = 1 \text{ and } m = n.$$

$$h < -|h_c|, \qquad S^z = -1 \text{ and } m = -n$$

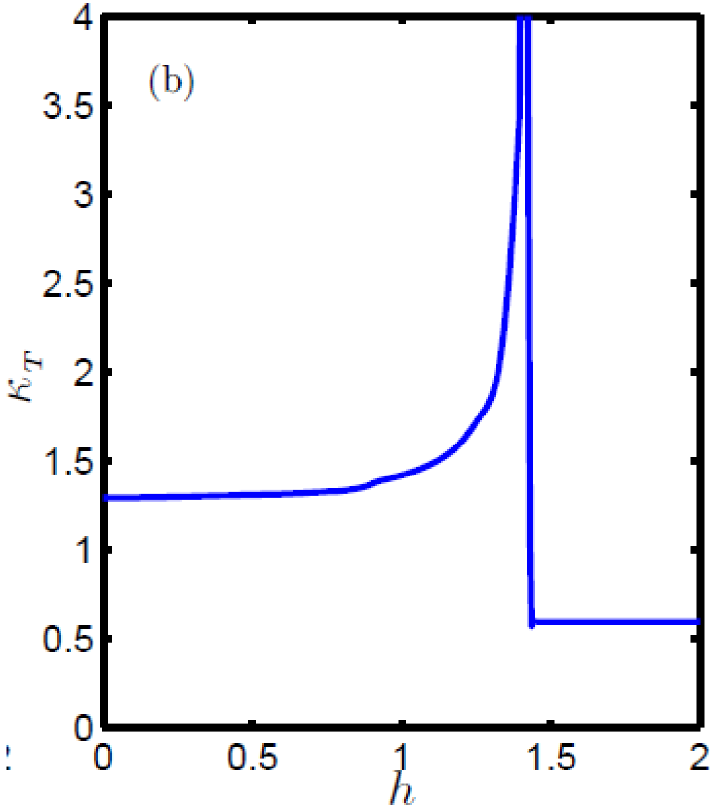
partially polarized states

Partially polarized states with $s^z=1$ (or -1) and $s^z=0$.

In the case of strong repulsive interaction, the critical field *hc* is

$$|h_c| = n^2\pi^2[1 - 16n/3\gamma + O(1/\gamma^2)] \qquad \gamma = c/n \gg 1$$

The compressibility at the ground state



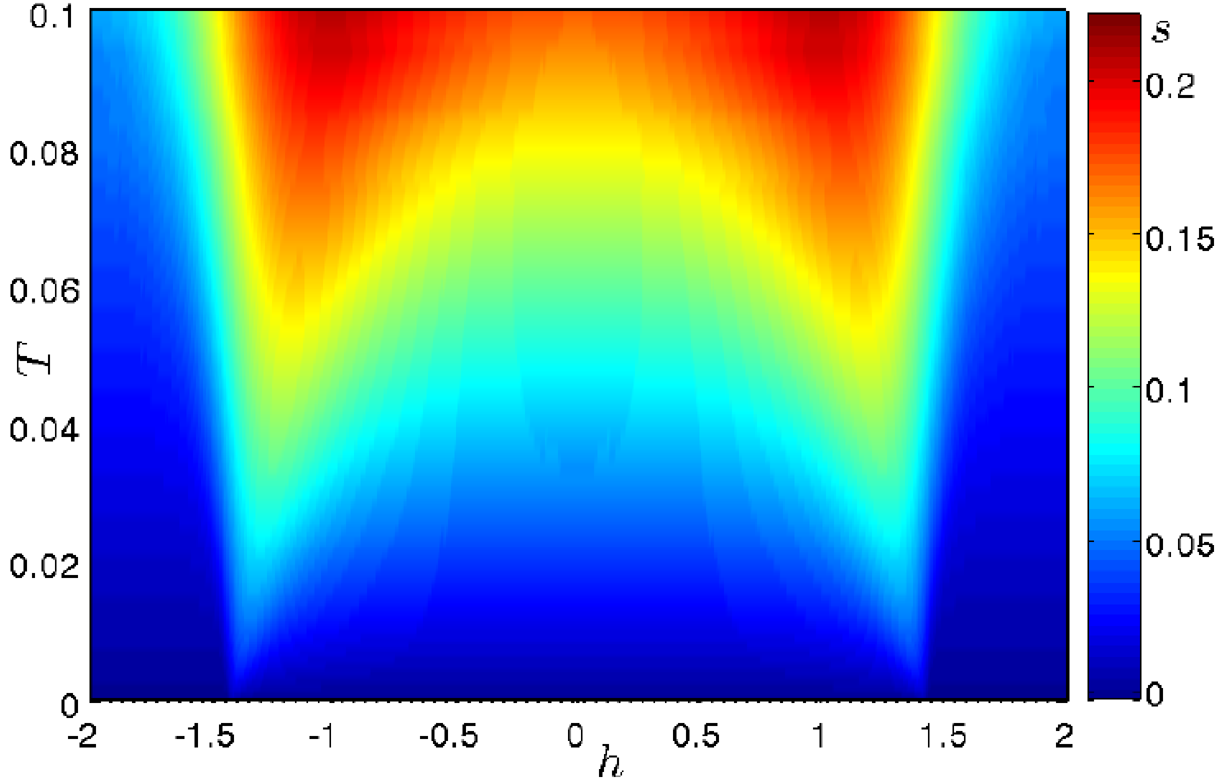
$n = 1$ and $c = 1$

$$\kappa_T = (\partial_\mu)^2 p|_{hT} / n^2$$

Compressibility can also be use to quantify the phase transition.

At the critical point, the compressibility is divergent.

Entropy of the model with different temperature and external magnetic field



$c = 1$ and $n = 1$

Scaling behavior near the critical point

Grand canonical ensemble

$$\mu_{c_1} = 0, \;\; \mu_{c_2} = -|h|,$$

when *u*<-|h|, the system is the vacuum state;
when -|h|<*u*<0 and h>0 (h<0), it is the fully upper (lower) polarized state;
when *u*>0, it is the partially polarized state.

Near critical point, density of particle number satisfies scaling law

$$n(\mu, h, T) - n_0(\mu, h, T) = T^{d/z+1-1/vz}\mathscr{F}\left(\frac{\mu - \mu_{c_1}}{T}\right)$$

where $d/z + 1 - 1/vz = 1/2$, $1/vz = 1$, and $\mathscr{F}(x) = \mathrm{F}_{-1/2}[(\mu - \mu_c)/T]/(2\pi^{1/2})$, dimension $d = 1$, critical exponent $z = 2$ and correlation length exponent $v = 1/2$.

3. Anisotropic spin-3/2 Fermi gas

Hamiltonian

$$\hat{H} = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \sum_{i\neq j}^{N} \sum_{lm} g_{lm} \hat{P}_{ij}^{lm} \delta(x_i - x_j) - \sum_{j=1}^{N} h \hat{f}_j^z$$

$$\hat{P}^{lm} = |lm\rangle\langle lm|.$$

No.	g_{00}	$g_{2,2}$	$g_{2,-1}$	$g_{2,0}$	$g_{2,1}$	$g_{2,2}$	
(i)	c	c	c	c	c	c	su(4)
(ii)	$3c$	c	c	c	c	c	so(5)
(iii)	c	$-c$	c	$-c$	c	$-c$	so(4)
	c	*c*	*-c*	*-c*	*-c*	*c*	

so(3)=su(2) ?

$$\hat{V}_{ab} = (\hat{P}_{ab}^{0,0} + \hat{P}_{ab}^{2,-1} + \hat{P}_{ab}^{2,1})c - (\hat{P}_{ab}^{2,-2} + \hat{P}_{ab}^{2,0} + \hat{P}_{ab}^{2,2})c$$

Symmetry

$$\hat{j}^{1,x} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \hat{j}^{1,y} = \frac{1}{2}\begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \end{pmatrix}, \quad \hat{j}^{1,z} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$\hat{j}^{2,x} = \frac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \hat{j}^{2,y} = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \quad \hat{j}^{2,z} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$|2,\ 2\rangle \quad |2,\ 0\rangle \quad |2,-2\rangle$$

$$\vec{\hat{J}}^{1} \times \vec{\hat{J}}^{1} = \mathrm{i}\vec{\hat{J}}^{1}. \qquad \vec{\hat{J}}^{2} \times \vec{\hat{J}}^{2} = \mathrm{i}\vec{\hat{J}}^{2}.$$

$$\downarrow \qquad \downarrow$$

$$SU(2) \otimes SU(2)$$

$$= SO(4)$$

$$|2,\ 1\rangle \quad |0,\ 0\rangle \quad |2,-1\rangle$$

SU(2) algebra

$$|1\rangle^{(1)} = -|2,2\rangle, |0\rangle^{(1)} = |2,0\rangle, |-1\rangle^{(1)} = |2,-2\rangle$$

$$\hat{j}^{1,\pm}|\mp 1\rangle^{(1)} = \sqrt{2}|0\rangle^{(1)}, \hat{j}^{1,\pm}|0\rangle^{(1)} = \sqrt{2}|\pm 1\rangle^{(1)}, \hat{j}^{1,\pm}|\pm 1\rangle^{(1)} = 0.$$

SU(2) algebra

$$|1\rangle^{(2)} = -|2,1\rangle, |0\rangle^{(2)} = |0,0\rangle, |-1\rangle^{(2)} = |2,-1\rangle$$

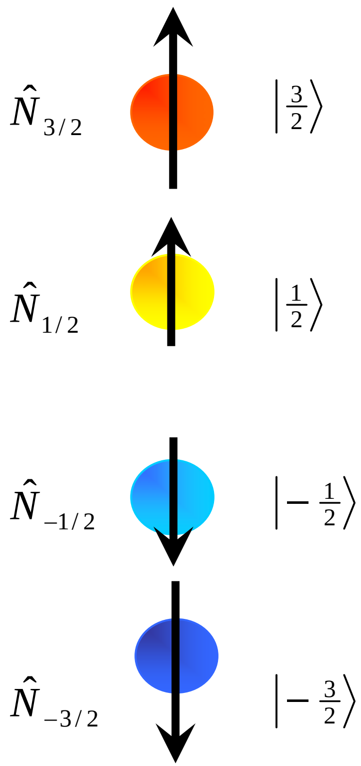
$$\hat{j}^{2,\pm}|\mp 1\rangle^{(2)} = \sqrt{2}|0\rangle^{(2)}, \hat{j}^{2,\pm}|0\rangle^{(2)} = \sqrt{2}|\pm 1\rangle^{(2)}, \hat{j}^{2,\pm}|\pm 1\rangle^{(2)} = 0.$$

These generators commute with the Hamiltonian.

$$[\hat{V}_{ij}, \hat{j}^{d\alpha}] = 0,\ d = 1,2,\ \alpha = x,y,z,$$

Conserved quantities

For the su(4) case, the number of atoms with spin 3/2, the number of atoms with spin 1/2, the number of atoms with spin -1/2, and the number of atoms with spin -3/2 are all conserved.



However, for the present case, the difference between number of atoms with spin 3/2 and number of atoms with spin -3/2 is conserved, and the difference between number of atoms with the spin half and that of atoms with spin minus half is conserved.

$$\hat{J}_{3/2} = \hat{J}^{1z} + \hat{J}^{2z} = \hat{N}_{3/2} - \hat{N}_{-3/2},$$

$$\hat{J}_{1/2} = \hat{J}^{1z} - \hat{J}^{2z} = \hat{N}_{1/2} - \hat{N}_{-1/2},$$

Wave function

$$\Psi_E = \sum_{\mathscr{P},\mathscr{Q}} \Theta(\mathscr{Q}) A^{m_1,m_2,\cdots,m_N}(\mathscr{Q},\mathscr{P}) \mathrm{e}^{\mathrm{i}\sum_j k_{\mathscr{P}_j} x_{\mathscr{Q}_j}}$$

Scattering matrix

$$\hat{S}_{ab}(k) = \frac{k-\mathrm{i}c}{k+\mathrm{i}c}\left(\hat{P}_{ab}^{0,0} + \hat{P}_{ab}^{2,-1} + \hat{P}_{ab}^{2,1}\right) + \frac{k+\mathrm{i}c}{k-\mathrm{i}c}\left(\hat{P}_{ab}^{2,-2} + \hat{P}_{ab}^{2,0} + \hat{P}_{ab}^{2,2}\right) + \sum_{l=1,3}\sum_{m=-l}^{l} \hat{P}_{ab}^{lm}.$$

Yang-Baxter equation

$$\hat{S}_{ab}(\lambda)\hat{S}_{ac}(\lambda+\mu)\hat{S}_{bc}(\mu) = \hat{S}_{bc}(\mu)\hat{S}_{ac}(\lambda+\mu)\hat{S}_{ab}(\lambda).$$

Energy and momentum

$$E=\sum_{j=1}^{N}k_j^2-hM,\ \ K=\sum_{j=1}^{N}k_j.$$

Bethe ansatz equations

$$e^{\mathrm{i}k_iL}=\prod_{j=1}^{M_1}\frac{k_i-\lambda_j-\mathrm{i}\frac{|c|}{2}}{k_i-\lambda_j+\mathrm{i}\frac{|c|}{2}}\prod_{j=1}^{M_2}\frac{k_i-\mu_j+\mathrm{i}\frac{|c|}{2}}{k_i-\mu_j-\mathrm{i}\frac{|c|}{2}},\ i=1\cdots N,$$

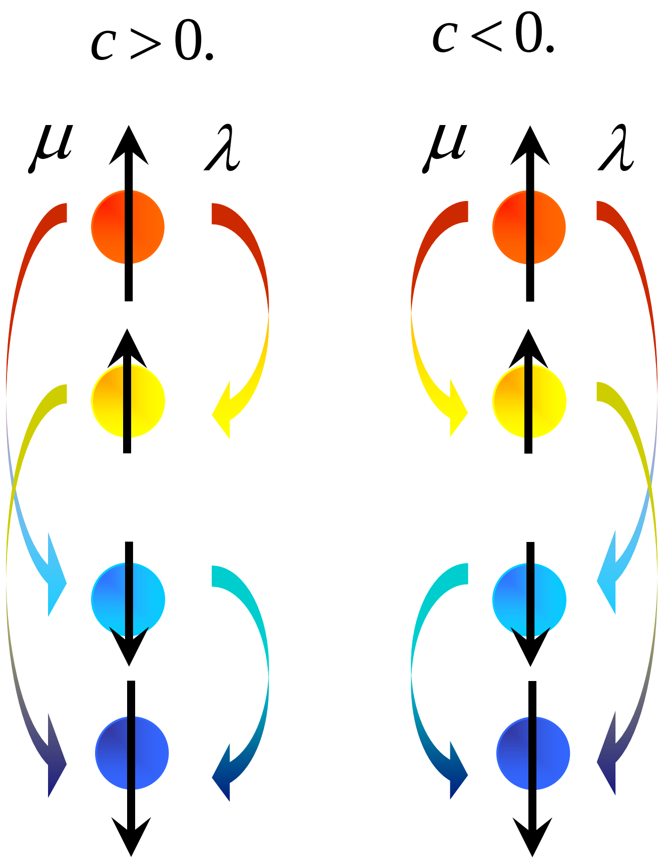
$$\prod_{i=1}^{N}\frac{\lambda_j-k_i-\mathrm{i}\frac{|c|}{2}}{\lambda_j-k_i+\mathrm{i}\frac{|c|}{2}}=\prod_{l\neq j}^{M_1}\frac{\lambda_j-\lambda_l-\mathrm{i}|c|}{\lambda_j-\lambda_l+\mathrm{i}|c|},\ j=1\cdots M_1,$$

$$\prod_{i=1}^{N}\frac{\mu_j-k_i-\mathrm{i}\frac{|c|}{2}}{\mu_j-k_i+\mathrm{i}\frac{|c|}{2}}=\prod_{l\neq j}^{M_2}\frac{\mu_j-\mu_l-\mathrm{i}|c|}{\mu_j-\mu_l+\mathrm{i}|c|},\ j=1\cdots M_2.$$

Magnetization

$$M=\begin{cases}3N/2-M_1-2M_2, & \text{when } c>0;\\ 3N/2-2M_1-M_2, & \text{when } c<0.\end{cases}$$

Explanations of the rapidities



	$c>0.$	$c<0.$
λ spin changing	-1	-2
μ spin changing	-2	-1

$$\hat{S}(k)=\begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&a_1&0&0&b_1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&a_2&0&0&0&0&0&b_2&0&0&0&0&0&0&0\\
0&0&0&p&0&0&z&0&0&-z&0&0&n&0&0&0\\
0&b_1&0&0&a_1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&z&0&0&p&0&0&n&0&0&-z&0&0&0\\
0&0&0&0&0&0&0&a_2&0&0&0&0&0&b_2&0&0\\
0&0&b_2&0&0&0&0&0&a_2&0&0&0&0&0&0&0\\
0&0&0&-z&0&0&n&0&0&p&0&0&z&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&a_1&0&b_1&0&0\\
0&0&0&p&0&0&-z&0&0&z&0&0&p&0&0&0\\
0&0&0&0&0&0&0&b_2&0&0&0&0&0&a_2&0&0\\
0&0&0&0&0&0&0&0&0&0&0&b_1&0&0&a_1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}.$$

|3/2> can not be directly flipped into |-3/2>, because there is no interaction in these two channels.

$$\hat{V}_{ab} = \left(\hat{P}_{ab}^{0,0} + \hat{P}_{ab}^{2,-1} + \hat{P}_{ab}^{2,1}\right)c - \left(\hat{P}_{ab}^{2,-2} + \hat{P}_{ab}^{2,0} + \hat{P}_{ab}^{2,2}\right)c =$$

$$\frac{1}{2}\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -c & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 \\
0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -c & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & -c & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Ground state

Solution of Bethe ansatz equations at T=0

k: 2 string & *u*: 2 string

$$k_{1,2} = \lambda \pm i|c|/2,$$
$$\mu_{1,2} = u \pm i|c|/2.$$

|2,2>, |2,-2> & |2,0> form a singlet pair

$|2,2\rangle|2,-2\rangle+|2,-2\rangle|2,2\rangle-|2,0\rangle|2,0\rangle/3^{1/2}$

Not the four particle spin-singlet state!

$$|1\rangle^{(1)} = -|2,2\rangle, |0\rangle^{(1)} = |2,0\rangle, |-1\rangle^{(1)} = |2,-2\rangle$$

$|1,-1\rangle + |-1,1\rangle - |0,0\rangle/3^{1/2}$ → Spin-1 singlet

Finite temperature

String hypothesis

$$k_{1,z} \in \mathbb{R}, z = 1,2,\cdots,N_1,;$$

$$k_{2,z,\pm} = k_{2,z} \pm \mathrm{i}c/2, \lambda_z^{k_2} = k_{2,z},$$

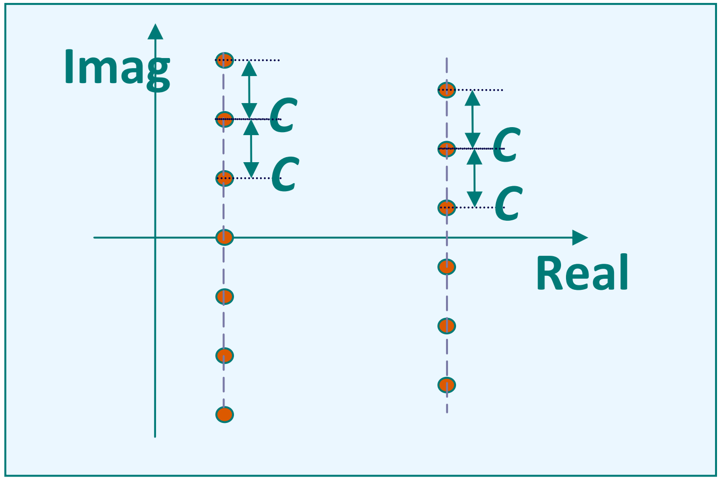
$$k_{2,z} \in \mathbb{R}, z = 1,2,\cdots,N_2;$$

$$\lambda_{n,z,j} = \lambda_{n,z} + (n+1-2j)\mathrm{i}c/2,$$

$$\lambda_{n,z} \in \mathbb{R}, j = 1,2,\cdots n, z = 1,2,\cdots,M_{1,n};$$

$$\mu_{n,z,j} = \mu_{n,z} + (n+1-2j)\mathrm{i}c/2,$$

$$\mu_{n,z} \in \mathbb{R}, j = 1,2,\cdots n, z = 1,2,\cdots,M_{2,n}.$$



Substituting the above string hypothesis into Bethe ansatz equations and taking the thermodynamic limit, we obtain following equations for the densities

$$\begin{aligned}
\rho_1^{\mathrm{h}}(k) &= \frac{1}{2\pi} - \rho_1(k) - \hat{a}_{\frac{1}{2}} * \rho_2 - \sum_m \hat{a}_{\frac{m}{2}} * \left[\rho_{1m}(k) - \rho_{2m}(k)\right], \\
\rho_2^{\mathrm{h}}(k) &= \frac{1}{\pi} - \rho_2(k) - \hat{a}_{\frac{1}{2}} * \rho_1(k) - \hat{a}_1 * \rho_2(k) + \sum_m \hat{A}'_{1m} * \rho_{2m}(k), \\
\rho_{1n}^{\mathrm{h}}(k) &= \hat{a}_{\frac{n}{2}} * \rho_1(k) - \sum_m \hat{A}_{nm} * \rho_{1m}(k), \qquad (60) \\
\rho_{2n}^{\mathrm{h}}(k) &= \hat{a}_{\frac{n}{2}} * \rho_1(k) + \hat{A}'_{1n} * \rho_2(k) - \sum_m \hat{A}_{nm} * \rho_{2m}(k).
\end{aligned}$$

where $*$ is a convolution operator defined by $f * g(k) = \int \mathrm{d} f(k-k')g(k)$ and convolution kernels are

$$a_n(k) = \frac{1}{\pi} \frac{nc}{k^2 + c^2}, \qquad (61)$$

$$\begin{aligned}
A_{nm}(k) &= a_{\frac{|n-m|}{2}} + 2a_{\frac{|n-m|+2}{2}} + \cdots + 2a_{\frac{n+m-2}{2}} + a_{\frac{n+m}{2}}, \\
A'_{nm}(k) &= A_{nm}(k) - \delta_{nm}\delta(k).
\end{aligned}$$

Consider the grand canonical ensemble, the Gibbs free energy is

$$
\begin{aligned}
g &= e - Ts - \mu n \\
&= \int \mathrm{d}k k^2 \rho_1(k) + \int \mathrm{d}k (2k^2 - \tfrac{c^2}{2}) \rho_2(k) \\
&- \mu \int \mathrm{d}k \rho_1(k) - 2\mu \int \mathrm{d}k \rho_2(k) \\
&- \tfrac{3}{2} h \int \mathrm{d}k \rho_1(k) - 2h \int \mathrm{d}k \rho_2(k) + h \textstyle\sum_{l=1}^{2} \sum_n n \int \mathrm{d}k \rho_{ln}(k) \\
&- T \int \mathrm{d}k \{\rho_1(k) \ln[1+\eta_1(k)] + \rho_1^{\mathrm{h}}(k) \ln[1+\eta_1(k)^{-1}]\} \\
&- T \int \mathrm{d}k \{\rho_2(k) \ln[1+\eta_2(k)] + \rho_2^{\mathrm{h}}(k) \ln[1+\eta_2(k)^{-1}]\} \\
&- T \textstyle\sum_{ln} \int \mathrm{d}k \{\rho_{ln}(k) \ln[1+\eta_{ln}(k)] + \rho_{ln}^{\mathrm{h}}(k) \ln[1+\eta_{ln}(k)^{-1}]\}
\end{aligned}
$$

Minimize the Gibbs free energy, we obtain the TBA

$$
\begin{aligned}
T \ln \eta_1(k) &= k^2 - \tfrac{3}{2} h - \mu + \hat{a}_{\frac{1}{2}} * T \ln[1+\eta_2(k)^{-1}] \\
&\quad - \textstyle\sum_n \hat{a}_{\frac{n}{2}} * \{T \ln[1+\eta_{1n}(k)^{-1}] + T \ln[1+\eta_{2n}(k)^{-1}]\} \\
T \ln \eta_2(k) &= 2k^2 - \tfrac{c^2}{2} - 2\mu - 2h + \hat{a}_{\frac{1}{2}} * T \ln[1+\eta_1(k)^{-1}] \\
&\quad + \hat{a}_1 * T \ln[1+\eta_2(k)^{-1}] - \textstyle\sum_n \hat{A}'_{1n} * T \ln[1+\eta_{2n}(k)^{-1}] \\
T \ln[1+\eta_{1n}(k)] &= nh + \hat{a}_{\frac{n}{2}} * T \ln[1+\eta_1(k)^{-1}] \\
&\quad + \textstyle\sum_m \hat{A}_{mn} * T \ln[1+\eta_{1m}(k)^{-1}] \\
T \ln[1+\eta_{2n}(k)] &= nh - \hat{a}_{\frac{n}{2}} * T \ln[1+\eta_1(k)^{-1}] \\
&\quad - \hat{A}'_{1n} * T \ln[1+\eta_2(k)^{-1}] + \textstyle\sum_m \hat{A}_{mn} * T \ln[1+\eta_{2m}(k)^{-1}]
\end{aligned}
$$

Define the dressed energy as

$$\varepsilon(k) = T \ln\left[\rho^{\mathrm{h}}(k)/\rho(k)\right],$$

Thermodynamic Bethe ansatz equations

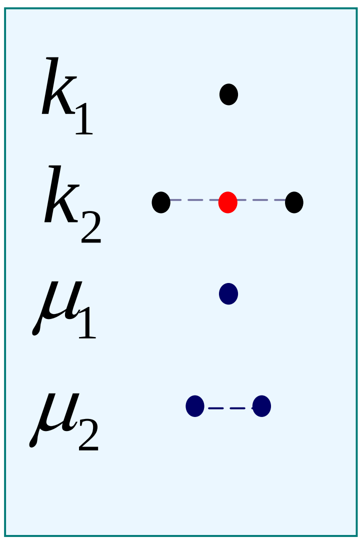
$$\begin{aligned}
&\varepsilon_1(k) = k^2 - \mu - \hat{G} * \varepsilon_2^-(k) + \hat{G} * \varepsilon_{1,1}^-(k) + \hat{G} * \varepsilon_{2,1}^-(k), \\
&\varepsilon_2(k) = 2k^2 - 2(c^2/4 + \mu) - \varepsilon_{2,1}(k), \\
&\varepsilon_{d,1}(k) = \hat{G} * \varepsilon_{d,2}^+(k) + (-1)^d \hat{G} * \varepsilon_1^-(k), \qquad (16) \\
&\varepsilon_{d,n}(k) = \hat{G} * \left[\varepsilon_{d,n-1}^+(k) + \varepsilon_{d,n+1}^+(k) + \delta_{d,2}\delta_{n,2}\varepsilon_2^-(k)\right], n > 1, \\
&\lim_{n\to\infty} \varepsilon_{1,n}(k)/n = h_1', \quad \lim_{n\to\infty} \varepsilon_{2,n}(k)/n = h_2',
\end{aligned}$$

where $\varepsilon_1(k)$, $\varepsilon_2(k)$, $\varepsilon_{1,n}(k)$ and $\varepsilon_{2,n}(k)$ are the dressed energies for 1-strings of k, 2-strings of k, n-strings of λ and n-strings of μ respectively. $\varepsilon^+(k) = T\ln[1 + \rho^{\mathrm{h}}(k)/\rho(k)]$ and $\varepsilon^-(k) = -T\ln[1 + \rho(k)/\rho^{\mathrm{h}}(k)]$. $*$ is a convolution operator defined by $\hat{G} * f(k) = \int G(k-k')f(k')\mathrm{d}k'$ and convolution kernel $G(k) = \mathrm{sech}(\pi k/c)/2c$. The renormalized magnetic field $h_1' = h, h_2' = 2h$, when $c > 0$, and $h_1' = 2h, h_2' = h$, when $c < 0$.

Ground state

h = 0 k: 2 string & *u*: 2 string

h is not zero



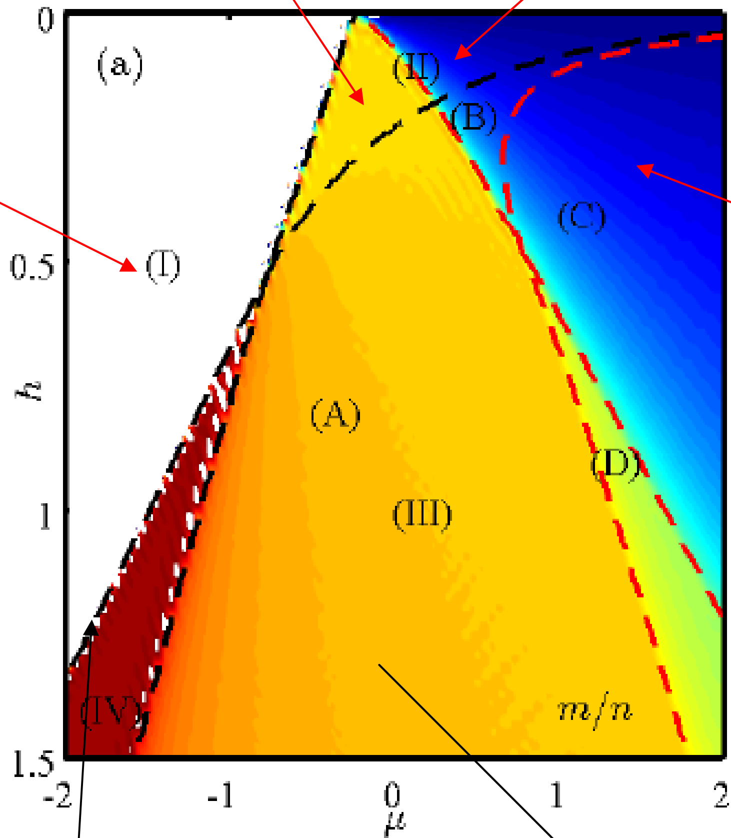
$c>0.$	$c<0.$
$c \quad \lvert 0,\ 0\rangle$	$c \quad \lvert 0,\ 0\rangle$
$-c \quad \lvert 2,\ 2\rangle$	$-c \quad \lvert 2,\ 2\rangle$
$c \quad \lvert 2,\ 1\rangle$	$c \quad \lvert 2,\ 1\rangle$
$-c \quad \lvert 2,\ 0\rangle$	$-c \quad \lvert 2,\ 0\rangle$
$c \quad \lvert 2,-1\rangle$	$c \quad \lvert 2,-1\rangle$
$-c \quad \lvert 2,-2\rangle$	$-c \quad \lvert 2,-2\rangle$

Phase diagram: c>0

Magnetic pair
|2,2>

k: 2-string,
|2,2>, |2,-2> &|2,0> : singlet pair

Vacuum



(B) (C) (D)

k: real & 2-string,
|2,2>,
|2,-2>,
|2,0>,
|3/2>.

Fully polarized state
|3/2>

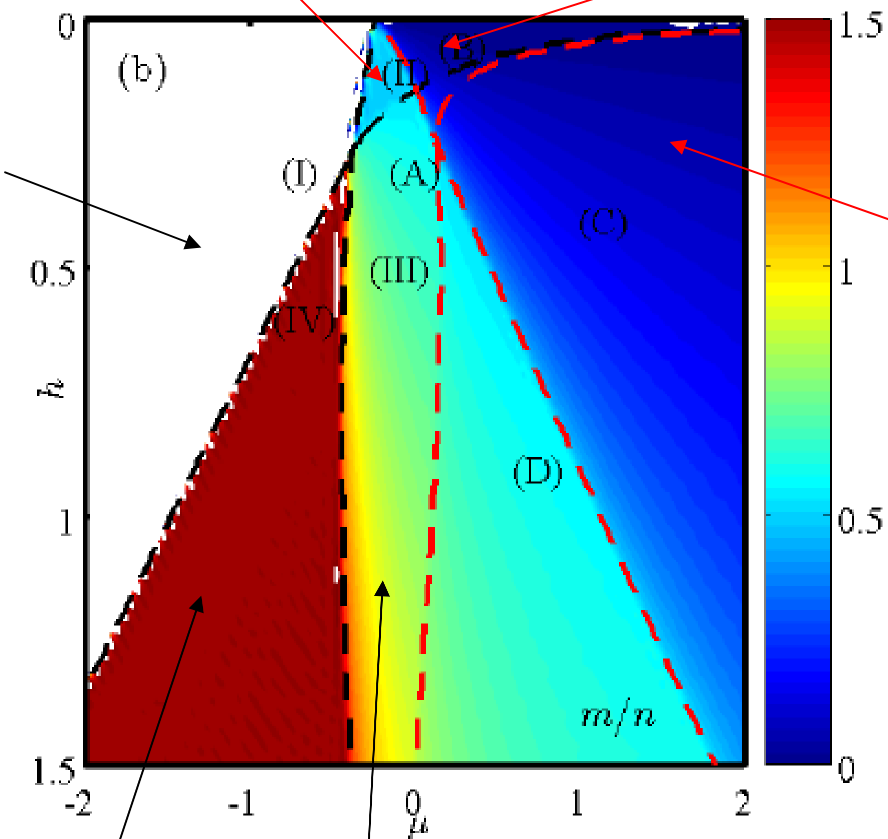
Partially polarized state
|3/2> & |2,2>

Phase diagram: c<0

Magnetic pair
|2,1>

k: 2-string,
|2,1>, |2,-1> & |0,0> : singlet pair

Vacuum



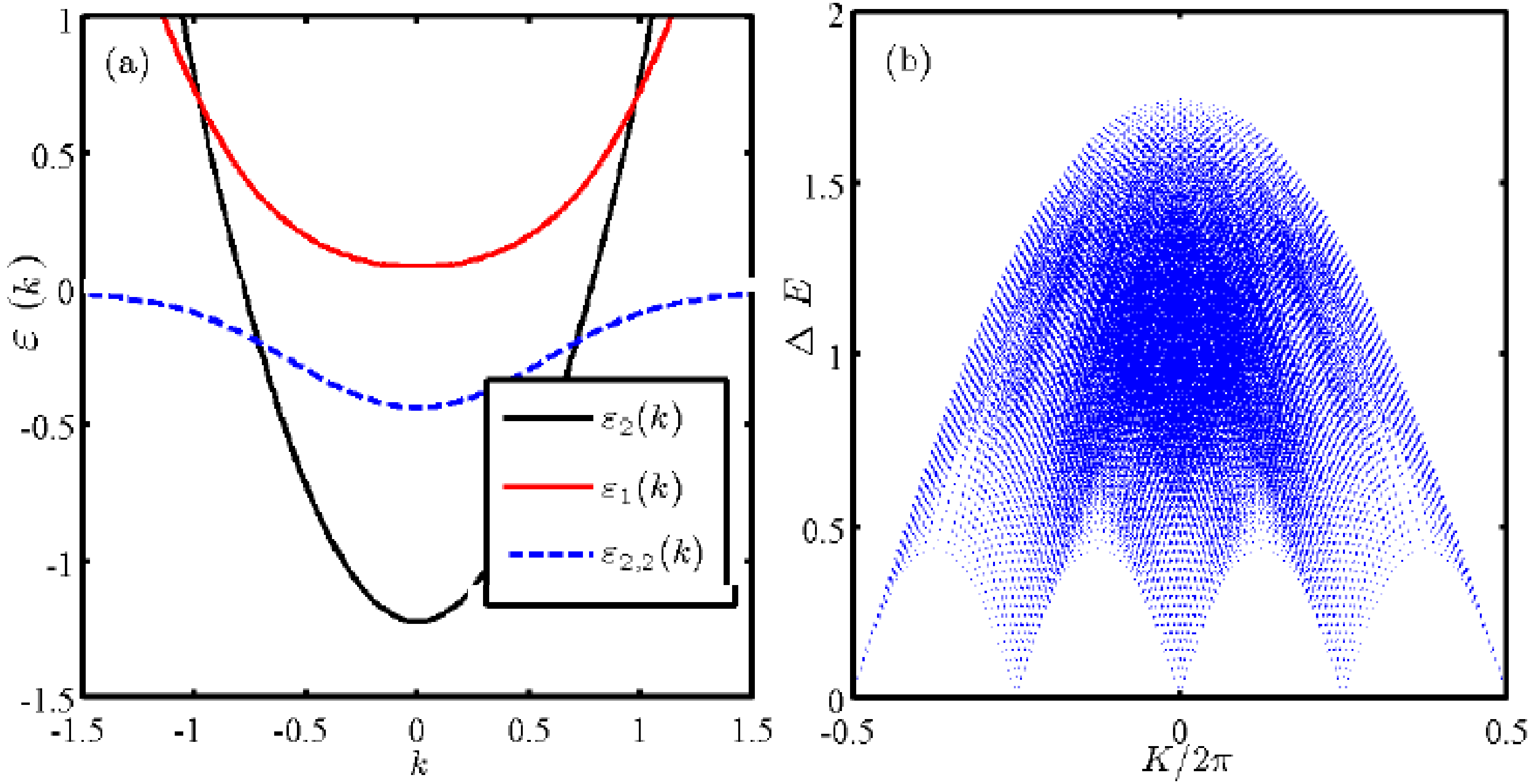
(B) (C) (D)

k: real & 2-string,
|2,1>,
|2,-1>,
|0,0>,
|3/2>.

Fully polarized state
|3/2>

Partially polarized state
|3/2> & |2,1>

Elementary excitation (h=0)



$$\Delta E = -\sum_{i=1}^{\eta} \varepsilon_{2,2}(\mu_j^{\mathrm{h}}), K = -\sum_{i=1}^{\eta} 2\pi \int_0^{\mu_j^{\mathrm{h}}} \rho_{2,2}(\mu)\mathrm{d}\mu,$$

Conclusion of part I

- ✓ By tuning the scattering strength in different channels, we can obtain the cold atoms with anisotropic spin-exchanging interaction.

- ✓ The anisotropic cold atom systems, especially for the large spin or multi-component ones, may have some interesting physics such as hidden symmetry, pairing mechanism and new quantum states.

II. Exactly solvable models with spin-orbital coupling

- ✓ This is a on-going project. In fact, it is only at the starting stage.
- ✓ We only construct the integrable model via mathematical analysis. The corresponding physical properties of the system is not studied.

Gauge field and spin-orbital coupling

Prospect from gauge field and SOC

1. Integer quantum Hall effect & Spin Hall effect
2. Topological phase transitions
3. Topological insulator & topological superfluid
4. Majorana Fermions

Experimental progress

1. Spielman (NIST)
   - Synthetic gauge field (2009,2011)
   - Spin-orbit coupled BEC (2011)
2. Chen (UTSC) : Spin-orbit coupling Bose gases (2012)
3. Zhang (Shanxi U) : Spin-orbit coupled Fermi gases (2012)
4. Zwierlein (MIT) : Spin-orbit coupled Fermi gases (2012)

Spin-orbit coupled Fermi gases

1. Tc enhancement [Yu and Zhai, PRL107, 195305 (2011)]
2. Imbalanced system [Iskin , PRL107, 050402 (2011)]
3. Mixed singlet-triplet superfluid [Hu et al., PRL107, 195304 (2011)]
4. Topological phase transition [Zhang et al., PRL107, 195304 (2011)]
5. Topology of tri-critical point [Liao et al., PRL108, 080406 (2012)]

Spin-orbit coupled Bose gases

1. Mean-field ground state: (Striped phase) [Ho and Zhang, PRL 107,150403 (2011); Li, Pitaevskii and Stringary, PRL 108, 225301 (2012)]
2. Collective excitation: [Zheng and Li, PRA 053607 (2012)]
3. Homogeneous system : (striped phase and plane wave phase) [Zhai, PRL 105, 160403 (2010)]
4. Trapped system: half-vortex [Stanescu, PRA2011, Santos, PRL2011, HuPRL2012…]

Beyond mean-field theory:

(a) Effective field theory: BKT transition , Zhai, PRB2011

(b) RG and T-matrix: Goldbart, PRA2012, Ozawa, PRA2012

(c) Bogoliubov: Barnett, PRA2012 , Ozawa, PRL2012, Cui, 2012…

(d) Hydrodynamic approach: Xu, PRL 2011, Xu, PRL2012

Hamiltonian

$$H = \sum_{\alpha} \int dx \psi_{\alpha}^{\dagger}(-\frac{\partial^2}{\partial x^2})\psi_{\alpha} + g \int dx(\psi_{\uparrow}^{\dagger}(-i\frac{\partial}{\partial x})\psi_{\uparrow} + \psi_{\downarrow}^{\dagger}(i\frac{\partial}{\partial x})\psi_{\downarrow}) + h \int dx(\psi_{\uparrow}^{\dagger}\psi_{\downarrow} + \psi_{\downarrow}^{\dagger}\psi_{\uparrow})$$
$$+ U \int dx \psi_{\alpha}^{\dagger}(x)\psi_{\beta}^{\dagger}(x)\psi_{\beta}(x)\psi_{\alpha}(x)$$

1. Two-component bosons with pseudo-spin up and down.
   All the interactions (same and different components) are equal.

2. The particle numbers of pseudo-spin-up and down are not conserved.
   The total number of particle is conserved.

3. If the transverse field is zero, the system can be solved exactly. The spin-orbital coupling can be gotten rid of by taking the gauge transformation. The system is equivalent to the interacting bosons with the twist boundary condition but without the spin-orbital coupling.

Step 1: consider the system without interaction

Take the Fourier transformation

$$\psi_\alpha = (\frac{1}{L})^{\frac{1}{2}} \sum_k e^{ikx} c_{k\alpha}$$

In the momentum space, we have

$$H_0 = \sum_k (\frac{\hbar^2 k^2 + gk}{2m} c^\dagger_{k\uparrow} c_{k\uparrow} + \frac{\hbar^2 k^2 - gk}{2m} c^\dagger_{k\downarrow} c_{k\downarrow}) + \sum_k h(c^\dagger_{k\uparrow} c_{k\downarrow} + c^\dagger_{k\downarrow} c_{k\uparrow})$$

Bogoliubov transformation

$$\begin{aligned}
a_{k_\uparrow} &= \cos\theta c_{k\uparrow} + \sin\theta c_{k\downarrow} \\
a_{k_\downarrow} &= -\sin\theta c_{k\uparrow} + \cos\theta c_{k\downarrow} \\
c_{k\uparrow} &= \cos\theta a_{k_\uparrow} - \sin\theta a_{k_\downarrow} \\
c_{k\downarrow} &= \sin\theta a_{k_\uparrow} + \cos\theta a_{k_\downarrow}
\end{aligned}$$

$$H_0 = \sum_k((\frac{\hbar^2k^2}{2m} + gk\cos 2\theta + h\sin 2\theta)a^\dagger_{k\uparrow}a_{k\uparrow} + (\frac{\hbar^2k^2}{2m} - gk\cos 2\theta - h\sin 2\theta)a^\dagger_{k\downarrow}a_{k\downarrow})$$
$$+ \sum_k((-gk\sin 2\theta + h\cos 2\theta)(a^\dagger_{k\uparrow}a_{k\downarrow} + a^\dagger_{k\downarrow}a_{k\uparrow}))$$

To eliminate the nondiagonal terms, we require

$$\tan 2\theta = \frac{h}{gk}$$

⇨

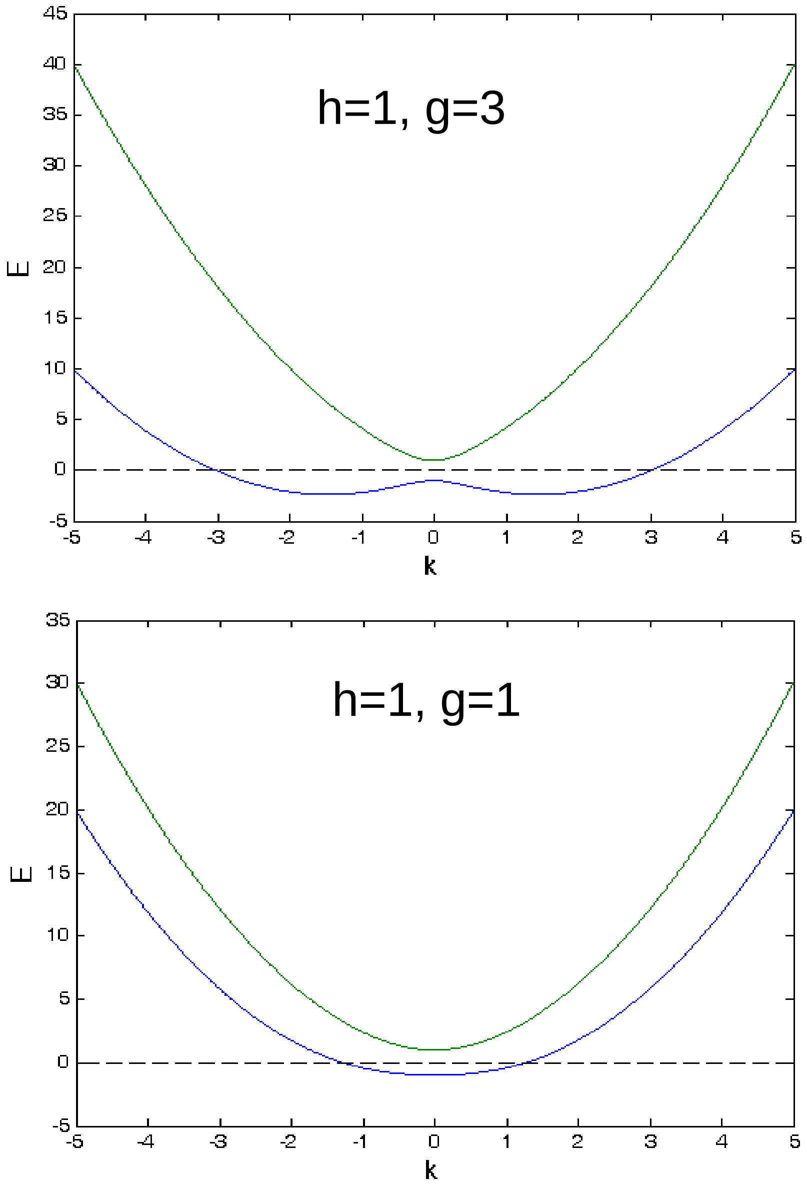
Depend on:
Transverse field
Spin-orbital coupling
Quasi-moment

⇨

The model parameters are free,
but the angle theta is constrained.

$$\begin{aligned} H_0 &= \sum_k(\frac{\hbar^2k^2}{2m} + gk\cos 2\theta + h\sin 2\theta)a^\dagger_{k\uparrow}a_{k\uparrow} + \sum_k(\frac{\hbar^2k^2}{2m} - gk\cos 2\theta - h\sin 2\theta)a^\dagger_{k\downarrow}a_{k\downarrow} \\ &= \sum_k(\frac{\hbar^2k^2}{2m} + \sqrt{h^2 + (gk)^2})a^\dagger_{k\uparrow}a_{k\uparrow} + \sum_k(\frac{\hbar^2k^2}{2m} - \sqrt{h^2 + (gk)^2})a^\dagger_{k\downarrow}a_{k\downarrow} \end{aligned}$$

The conditions of integrability



1. We focus on the lower band.

2. For different values of *h* and *g*, the dispersion relations of the lower band are different.

3. When the lower band is quadratic, that is to say, if there does not exist the case that four quasi-momentum are equal, the model can be solved exactly. This means that the transverse field in the system should be large enough compare to the spin-orbital coupling.

large h & small g : $h > g^2/2$
critical value : $h = g^2/2$

Step 2: consider the two-body scattering with interaction

Assume the two-body wave-function takes the form of

phase shift

$$|k_1,k_2\rangle = \int dx_1 dx_2 \{\exp(ik_1x_1)\exp(ik_2x_2)[1 + i\lambda\epsilon(x_1 - x_2)]$$
$$+ \exp(ik_2x_1)\exp(ik_1x_2)[1 - i\lambda\epsilon(x_1 - x_2)]\}\psi^\dagger(x_1,k_1)\psi^\dagger(x_2,k_2)|0\rangle$$

Where

$$\psi(x_i,k_i) = -\sin\theta_i\psi_\uparrow(x_i) + \cos\theta_i\psi_\downarrow(x_i)$$

From

$$\epsilon(x_1 - x_2) = \theta(x_1 - x_2) - \theta(x_2 - x_1)$$

We have

$$\frac{\partial}{\partial x_1}\epsilon(x_1 - x_2) = 2\delta(x_1 - x_2),\ \frac{\partial}{\partial x_2}\epsilon(x_1 - x_2) = -2\delta(x_2 - x_1)$$

The Hamiltonian without interaction acting on the assumed state gives two kinds of terms.

$$H_0|k_1,k_2\rangle = E(k_1,k_2)|k_1,k_2\rangle + |R\rangle$$

unwanted term

eigenvalues

$$\int dx_1 dx_2 \{[-((k_1^2+gk_1)\chi_1 \sin\theta_1 - h\chi_1\cos\theta_1)\psi_\uparrow^\dagger(x_1)$$
$$+((k_1^2-gk_1)\chi_1\cos\theta_1 - h\chi_1\sin\theta_1)\psi_\downarrow^\dagger(x_1)]\psi^\dagger(x_2,k_2) + (x_1,k_1 \longleftrightarrow x_2,k_2) + (k_1 \leftrightarrow k_2)\}|0\rangle$$
$$= (E_{k_1}+E_{k_2})|k_1,k_2\rangle$$

1. This term does not contain $\delta(x_1 - x_2)$. That is the two particle have the different positions.

   This term gives the eigenvalues.

2. Using $\tan 2\theta = \frac{h}{gk}$, it is easy to show that

$$(k^2-gk)\cos\theta - h\sin\theta = E_k\cos\theta$$
$$(k^2+gk)\sin\theta - h\cos\theta = E_k\sin\theta$$

The unwanted term is

$$|R\rangle = \int dx 2\lambda \chi(x,x)(k_1 - k_2)[(\psi_\uparrow^\dagger(x)\psi_\uparrow^\dagger(x)\sin\theta_1 \sin\theta_2 + \psi_\downarrow^\dagger(x)\psi_\downarrow^\dagger(x)\cos\theta_1 \cos\theta_2) - \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\sin(\theta_1 + \theta_2)]\,|0\rangle$$

This term contain the delta-function $\delta(x_1 - x_2)$. That is to say, if the two particle are in the same position, this term has the contribution.

The contribution of this term is giving the possible values of the quasi-momentum.

Then we consider the interaction. The interaction acting on the wave-function gives

$$U|k_1,k_2\rangle = |R'\rangle$$

$$|R'\rangle = \int dx 2U\chi(x,x)[(\psi_\uparrow^\dagger(x)\psi_\uparrow^\dagger(x)\sin\theta_1\sin\theta_2 + \psi_\downarrow^\dagger(x)\psi_\downarrow^\dagger(x)\cos\theta_1\cos\theta_2) - \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\sin(\theta_1+\theta_2)]|0\rangle$$

If the assumed wave-function is the eigenstate of the system, that is if the following Schrodinger eigen-equation is satisfied,

$$H|k_1,k_2\rangle = (E_{k_1} + E_{k_2})|k_1,k_2\rangle$$

we must require the unwanted terms are cancelled with each other.

$$|R\rangle + |R'\rangle = 0$$

This condition gives the possible values of undetermined parameter lambda

$$\lambda = -\frac{U}{k_1 - k_2}$$

Then the two-body scattering phase shift is

$$\boxed{\frac{1+i\lambda}{1-i\lambda} = \exp(i\phi(k_1,k_2))}$$

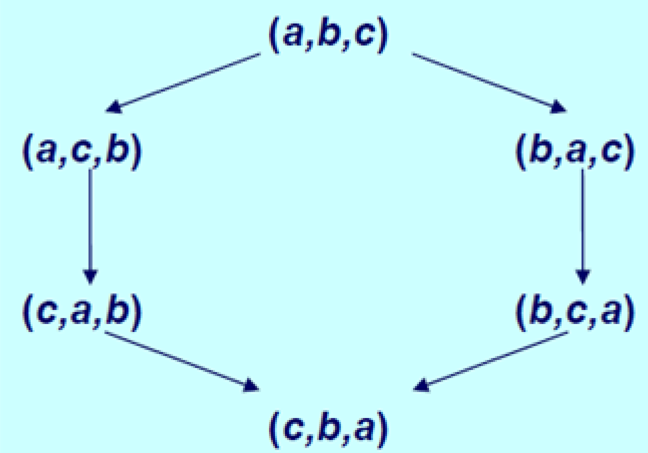
Then we obtain the two-body scattering matrix as

$$S = \frac{k_1 - k_2 - iU}{k_1 - k_2 + iU}$$

We see that the two-body scattering matrix is a number instead of a matrix.

It satisfies the Yang-Baxter equation. Thus the system is integrable.

$$\hat{S}_{ab}(\lambda)\hat{S}_{ac}(\lambda+\mu)\hat{S}_{bc}(\mu) = \hat{S}_{bc}(\mu)\hat{S}_{ac}(\lambda+\mu)\hat{S}_{ab}(\lambda).$$



$$S_{ab}(k_b - k_a)S_{ac}(k_c - k_a)S_{bc}(k_c - k_b)$$
$$= S_{bc}(k_c - k_b)S_{ac}(k_c - k_a)S_{ab}(k_b - k_a).$$

Completely integrable:

degrees of freedom = conversed quantities

Transfer matrix: constructed from the two-body scattering matrices.

$$[t(u_1), t(u_2)] = 0.$$

Step 3: consider the many-body problem

According to the coordinate Bethe ansatz method, we assume the many-body eigen-state as

$$|k_1, \cdots, k_M\rangle = \sum_{i=1}^{M} \sum_{x_i=1}^{N} \psi(x_1, \cdots, x_M) \psi_{x_1}^{-} \cdots \psi_{x_M}^{-} |0\rangle$$

Wave function

$$\psi(x_1, \cdots, x_M) = \sum_{P,Q} A_P e^{i \sum_{j=1}^{M} k_{P_j} x_{Q_j}} \theta(x_{Q_1} < \cdots < x_{Q_M})$$

The many-particle scattering process can be divided into the production of two-particle scattering.

$$S_{12\cdots N} = S_{1N} S_{1N-1} \cdots S_{12}$$

Periodic boundary condition

$$S_{jN} S_{jN-1} \cdots S_{jj+1} S_{jj-1} \cdots S_{j1} e^{ik_j L} \xi_0 = \xi_0.$$

Bethe ansatz equations

$$e^{ik_j N} = \prod_{l \neq j}^{M} S_{jl}^{-1}, \quad j = 1, \cdots, M.$$

Conclusion

$$H = \sum_{\alpha} \int dx \psi_{\alpha}^{\dagger}(-\frac{\partial^2}{\partial x^2})\psi_{\alpha} + g \int dx(\psi_{\uparrow}^{\dagger}(-i\frac{\partial}{\partial x})\psi_{\uparrow} + \psi_{\downarrow}^{\dagger}(i\frac{\partial}{\partial x})\psi_{\downarrow}) + h \int dx(\psi_{\uparrow}^{\dagger}\psi_{\downarrow} + \psi_{\downarrow}^{\dagger}\psi_{\uparrow})$$
$$+ U \int dx \psi_{\alpha}^{\dagger}(x)\psi_{\beta}^{\dagger}(x)\psi_{\beta}(x)\psi_{\alpha}(x)$$

The energy spectrum of the system is

$$E = \sum_{k}(\frac{\hbar^2 k^2}{2m} - \sqrt{h^2 + (gk)^2})$$

where the quasi-momentum should satisfy the following Bethe ansatz equations.

$$e^{ik_j L} = -\prod_{l=1}^{N} \frac{k_j - k_l + iU}{k_j - k_l - iU}$$

The studies of the physics of the model are proceeding!

Thank you for your attention!