An exactly solvable pairing model with p + ip wave symmetry

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Overview

- Recent interest in p-wave superconductivity is motivated by several applications: He3 films, superconductors as Sr2RuO4, superfluids of fermi cold atoms in optical traps, etc.

- The BCS model with p+ip pairing symmetry gives rise to the so called pfaffian state. This state is a close relative of the Moore and Read state, which is one of the candidates for the ground state of the Fractional Quantum Hall effect at filling fraction 5/2. In fact, the BCS model provides a physical picture of the nature of this “exotic” state.

-When considering vortices in the BCS model one gets non abelians anyons similar to those of the Moore-Read model (Green-Read 2000). Thus the p+ip superconductors, as the 5/2 FQH state may allow for Topological Quantum Computation, although non universal.

So far the studies of the BCS model with p+ip symmetry are based on a mean-field analysis using the BdG Hamiltonian.
The corresponding phase diagram contains three regions:

- Weak coupling -> “standard” Cooper pairs (BCS type)
- Weak pairing -> “Moore-Read” pairs
- Strong pairing -> localized Cooper pairs (BEC type)

The weak and strong pairing regions are separated by a second order phase transition where the gap vanishes (Read-Green).
Moreover, the mean field wave function also experiences a “topological phase transition” across these two regions (Volovik).

The boundary between the weak pairing and the weak coupling regions has not been well characterized by mean field studies.

It is thus of great interest to have an exactly solvable BCS model with p+ip symmetry to analyze in detail the nature of the Moore-Read Pfaffian state and the different phases boundaries of the model.

This model is the so called “reduced” BCS model with p+ip wave symmetry and it is analogous to the reduced BCS model with s-wave symmetry. The latter model was solved by Richardson in 1963 and it is closely related to the Gaudin spin Hamiltonians.

The Richardson model was extensively used to study ultrasmall superconducting grains made of Al in the 1990s.

The integrability of the Richardson-Gaudin models can be proved using the standard Quantum Inverse Scattering Methods. These are the methods that we apply to the p+ip model.

Outline

- The pfaffian state in the BCS model

- Mean-field approach to the p + i p model

- Exact Bethe ansatz solution

- Numerical solution of the BAEs

- Electrostatic analogy of the BAEs

- Questions and suggestions

The pfaffian state

The BCS state (Proyected BCS state) for p-wave

$$|BCS\rangle \propto \exp\left(\frac{1}{2}\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)|0\rangle \rightarrow |PBCS\rangle \propto \left(\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)^N |0\rangle$$

$c^*_{\vec{k}}$ Operator that creates a polarised electron with momenta k

$g_{\vec{k}}$ (Odd) Wave function of the Cooper pair in momentum space

$$\phi(\vec{r}) = \sum_{\vec{k}} g(\vec{k})\exp(i\,\vec{k}\cdot\vec{r})$$ Wave function in real space

The proyected state has 2N electrons with a pfaffian wave function

$$\psi(\vec{r}_1,\ldots,\vec{r}_{2N}) = A\left[\phi(\vec{r}_1-\vec{r}_2)\,\phi(\vec{r}_3-\vec{r}_4)\ldots\phi(\vec{r}_{2N-1}-\vec{r}_{2N})\right]$$
$$= \sqrt{\det \phi(\vec{r}_i - \vec{r}_j)}$$

Moore-Read state corresponds to the long distance/small momenta behaviour:

$$\phi(\vec{r}) \propto 1/(x+i\,y),\ |\vec{r}| \rightarrow \infty, \quad g(\vec{k}) \propto 1/(k_x + i\,k_y),\ |\vec{k}| \rightarrow 0$$

BCS mean-field theory

The reduced BCS Hamiltonian with s-wave symmetry (Richardson model)

$$H = \sum_{\vec{k},\sigma} \frac{\vec{k}^2}{2m} c^*_{\vec{k},\sigma} c_{\vec{k},\sigma} - G \sum_{\vec{k} \neq \vec{k}'} c^*_{\vec{k},\uparrow} c^*_{-\vec{k},\downarrow} c_{-\vec{k}',\downarrow} c_{\vec{k}',\uparrow}$$

The reduced BCS Hamiltonian with p+ip wave symmetry reads:

$$H = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} c^*_{\vec{k}} c_{\vec{k}} - \frac{G}{4m} \sum_{\vec{k} \neq \vec{k}'} (k_x - i k_y)(k'_x + i k'_y) c^*_{\vec{k}} c^*_{-\vec{k}} c_{-\vec{k}'} c_{\vec{k}'}$$

In the mean-field approximation one defines the order parameter:

$$\Delta = \sum_{\vec{k}} (k_x + i k_y) \left\langle c_{-\vec{k}} c_{\vec{k}} \right\rangle$$

$$H = \sum_{\vec{k}} \left(\frac{\vec{k}^2}{2m} - \frac{\mu}{2} \right) c^*_{\vec{k}} c_{\vec{k}} - \frac{G}{2m} \sum_{k_x > 0, k_y} \left(\Delta (k_x - i k_y) c_{-\vec{k}'} c_{\vec{k}'} + h.c. \right)$$

Which can be diagonalized by a Bogoliubov transformation

The gap Δ and chemical potential μ are solution of the eqs (m=1)

$$\sum_{k_x \geq 0, k_y} \frac{\vec{k}^2}{\sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2}} = \frac{1}{G}$$

$$\mu \sum_{k_x \geq 0, k_y} \frac{1}{\sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2}} = 2N - L + \frac{1}{G}$$

L is the number of energy levels and N is the number of pairs

The thermodynamic limit is defined by

$$L \to \infty, \quad N \to \infty, \quad G \to 0$$

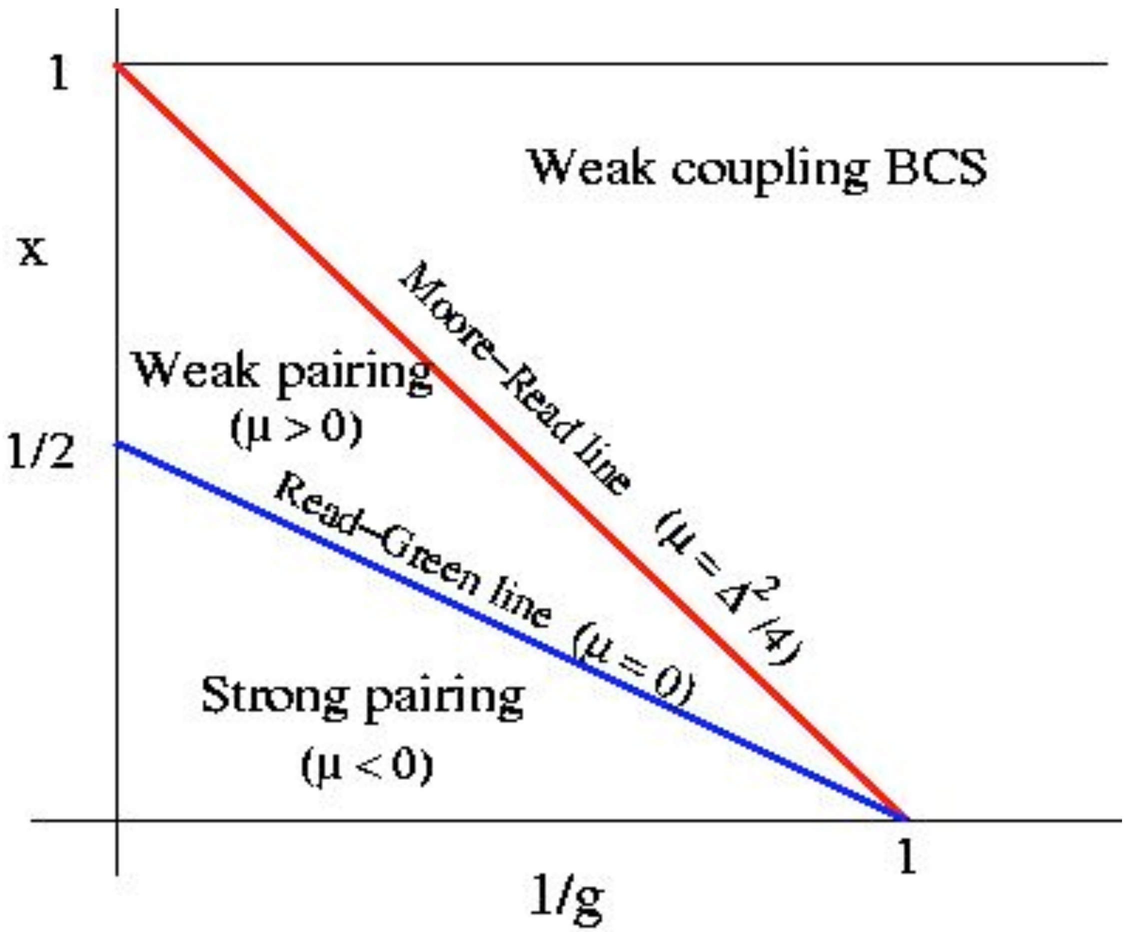
Such that $g = GL, \quad x = \frac{N}{L} (\textit{filling factor})$ are constant

The phase diagram is two dimensional:

$$0 < g < \infty, \quad 0 < x < 1$$

Phase diagram of the p + ip wave model

Combination of mean field + exact results



The mean field wave function:

$$|BCS\rangle \propto \exp\left(\frac{1}{2}\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)|0\rangle \rightarrow |PBCS\rangle \propto \left(\sum_{\vec{k}} g_{\vec{k}}\, c^*_{\vec{k}}\, c^*_{-\vec{k}}\right)^N |0\rangle$$

$$g(\vec{k}) = \frac{v_{\vec{k}}}{u_{\vec{k}}} = \frac{E(\vec{k}) - \vec{k}^2 + \mu}{\left(k_x + i\,k_y\right)\Delta^*}$$

where $E(\vec{k}) = \sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2|\Delta|^2}$ is the energy of the quasiparticles

The behaviour of $g(\vec{k})$ as k -> 0 depends crucially on the sign of μ

$$\mu > 0 \rightarrow g(\vec{k}) \approx \frac{1}{k_x + i\,k_y},\ \phi(\vec{r}) \approx \frac{1}{x + i\,y} : Weak\ pairing\ phase$$

$$\mu < 0 \rightarrow g(\vec{k}) \approx k_x - i\,k_y,\ \phi(\vec{r}) \approx e^{-r/r_0} : Strong\ pairing\ phase$$

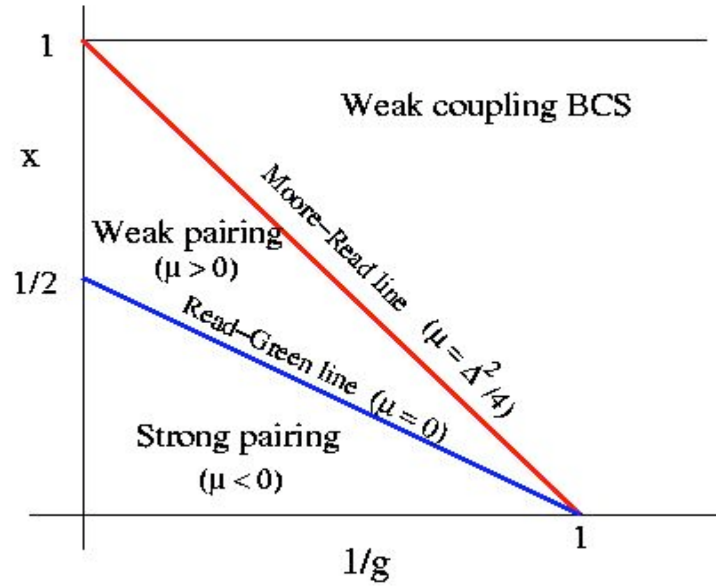
At $\mu = 0$ there is a second order phase transition (Read-Green line)

$$E(\vec{k}) \propto \left|\vec{k}\right| \rightarrow 0$$

Solution of the gap and chemical potential equations

Parameterize the dispersion relation

$$E(\vec{k}) = \sqrt{\left(\vec{k}^2 - \mu\right)^2 + \vec{k}^2 |\Delta|^2} = \sqrt{\left(\vec{k}^2 - a\right)\left(\vec{k}^2 - b\right)}$$



Weak coupling	$a,b = \varepsilon_0 \pm i\Delta_0$	$\mu > \Delta^2/4$	$x > x_{MR}$
Moore – Read line	$a = b = -\mu$	$\mu = \Delta^2/4$	$x_{MR} = \left(1 - \frac{1}{g}\right)$
Weak pairing	$a < b < 0$	$0 < \mu < \Delta^2/4$	$x_{RG} < x < x_{MR}$
Read – Green line	$a < b = 0$	$\mu = 0$	$x_{RG} = \frac{1}{2}\left(1 - \frac{1}{g}\right)$
Strong pairing	$a < b < 0$	$\mu < 0$	$x < x_{RG}$

a and b have a meaning in the electrostatic solution of the exact model

Duality between weak pairing and strong pairing phases

Given two points in the phase diagram

$$(g, x_I) \in weak\ \ pairing\ \ phase\ (\mu > 0)$$

$$(g, x_{II}) \in strong\ \ pairing\ \ phase\ (\mu < 0)$$

$$If \quad x_I + x_{II} = 1 - \frac{1}{g} \Rightarrow E_I = E_{II},\ \Delta_I = \Delta_{II}, \mu_I = -\mu_{II}$$

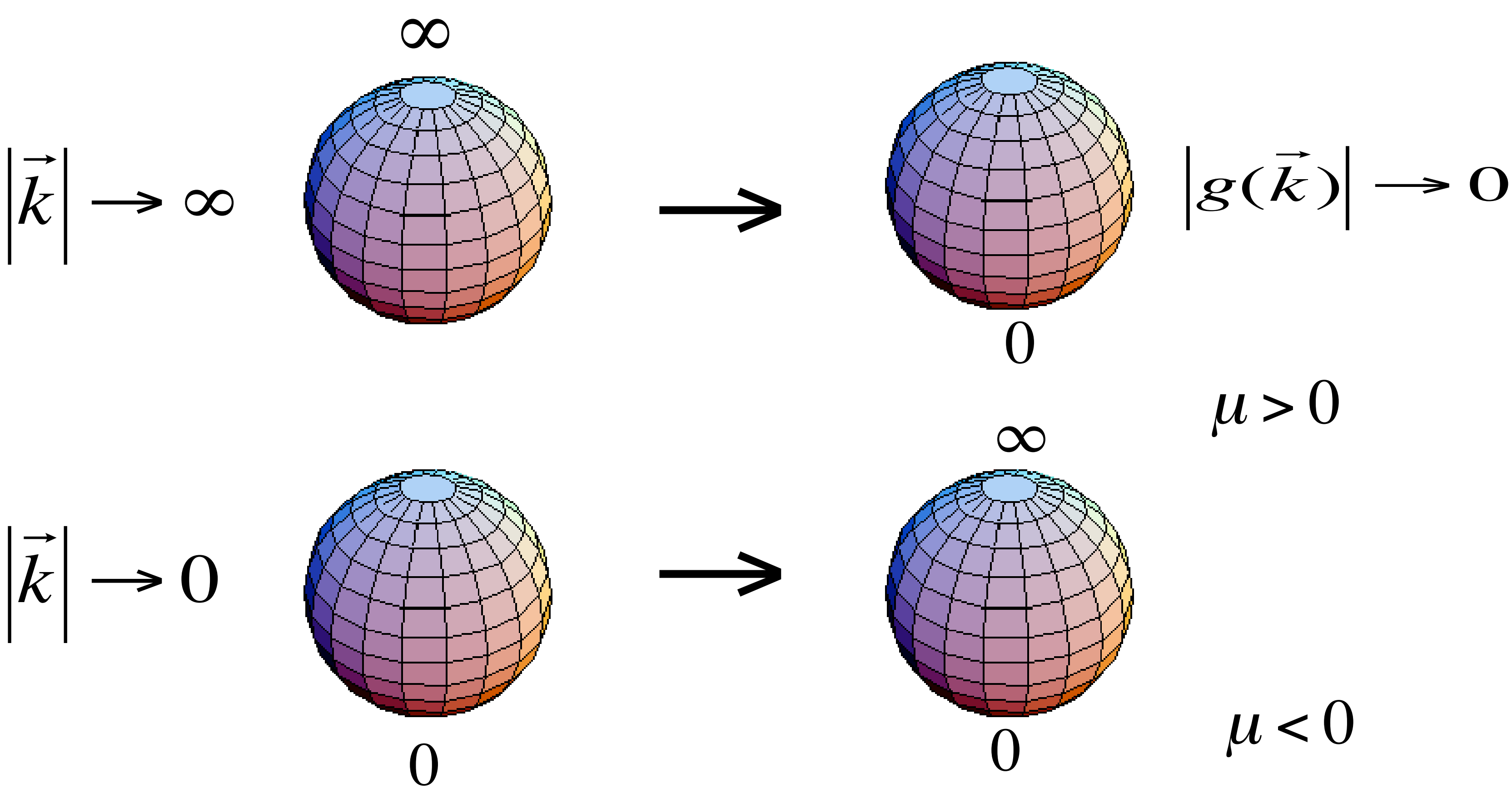
The Read-Green line is selfdual

The Moore-Read line is dual to the “empty” state x = 0
(in particular the GS energy on this line is zero)

This duality also appears in the exact solution and has an
Interesting interpretation

Mean field topological weak-strong transition at $\mu = 0$ (Volovik)

Momentum space | Wave function



Winding number of the map $S_2 \to S_2 : \begin{cases} +1 & \mu > 0 \ (weak\ pairing) \\ 0 & \mu < 0 \ (strong\ pairing) \end{cases}$

The Bethe ansatz solution

Recall the Hamiltonian:

$$H = \sum_{\vec{k}} \frac{\vec{k}^2}{2m} c_{\vec{k}}^* c_{\vec{k}} - \frac{G}{4m} \sum_{\vec{k} \neq \vec{k}'} (k_x - i k_y)(k'_x + i k'_y) c_{\vec{k}}^* c_{-\vec{k}}^* c_{-\vec{k}'} c_{\vec{k}'}$$

Setting $z_{\vec{k}}^2 = \vec{k}^2 / m$

Define the hard core boson operators: $b_{\vec{k}}^* = \frac{k_x - i k_y}{\left|\vec{k}\right|} c_{\vec{k}}^* c_{-\vec{k}}^*$

Then the Hamiltonian can be brought into the form

$$H = \sum_{k_x \geq 0, k_y} z_{\vec{k}}^2 b_{\vec{k}}^* b_{\vec{k}} - G \sum_{\vec{k} \neq \vec{k}'} z_{\vec{k}} z_{\vec{k}'} b_{\vec{k}}^* b_{\vec{k}'}$$

And can be solved using the Quantum Inverse Scattering Method starting from the XXZ R-matrix and taking a quasi-classical limit

The Schroedinger equation: $H|\psi\rangle = E|\psi\rangle$

is solved exactly by the states (m=1)

$$|\psi\rangle = \prod_{m=1}^{N} C(y_m)|0\rangle, \qquad C(y) = \sum_{k_x \geq 0, k_y} \frac{k_x - i k}{\vec{k}^2 - y} \, c^*_{\vec{k}} \, c^*_{-\vec{k}}$$

where the "rapidities" y_m satisfy the Bethe ansatz eqs

$$-\frac{G^{-1} - L + 2N - 1}{y_m} - \sum_{k=1}^{L} \frac{1/2}{y_m - z_k^2} + \sum_{j \neq m}^{N} \frac{1}{y_m - y_j} = 0 \quad (m = 1, \ldots, N)$$

The total energy is $E = (1+G)\sum_{m=1}^{N} y_m$

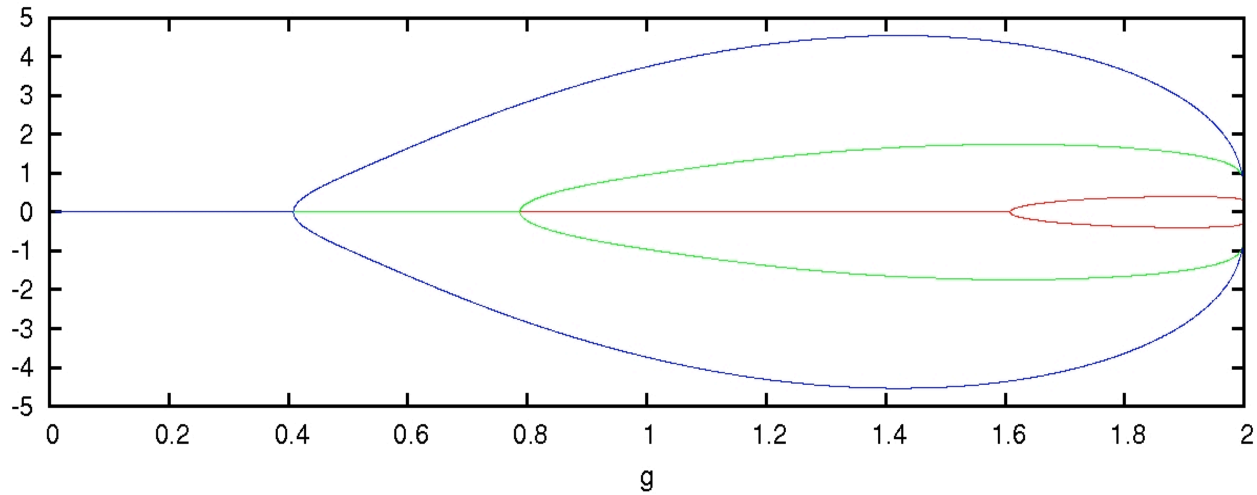
$$\lim_{G \to 0} y_m = z_k^2$$

$$For \; G \neq 0 \to y_m : real \; or \; complex$$

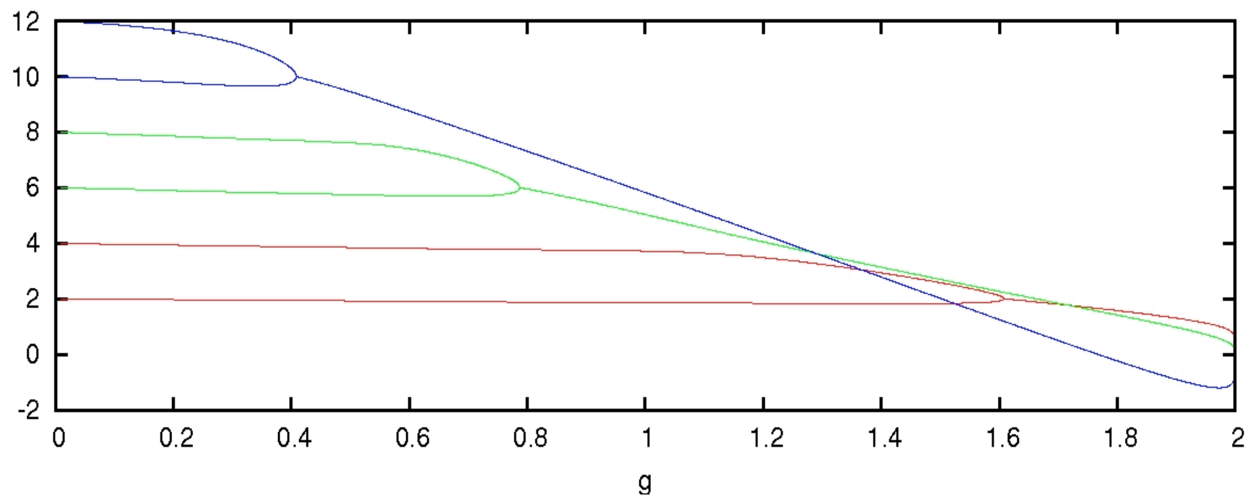
Complex solutions always appear in conjugate pairs

Roots of the p + i p model (numerical solution with L=12,N=6, x=1/2)

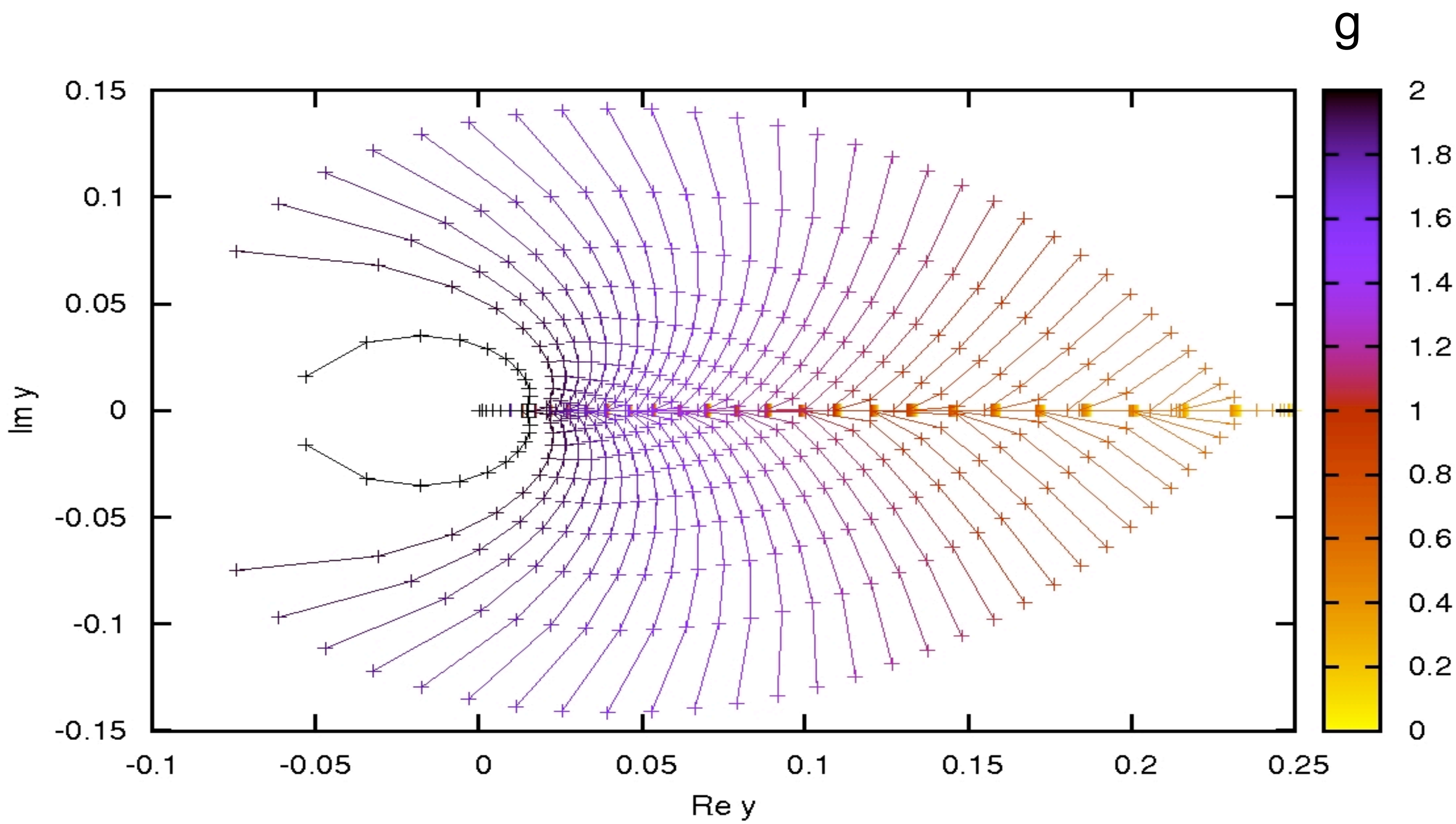
Im y_m



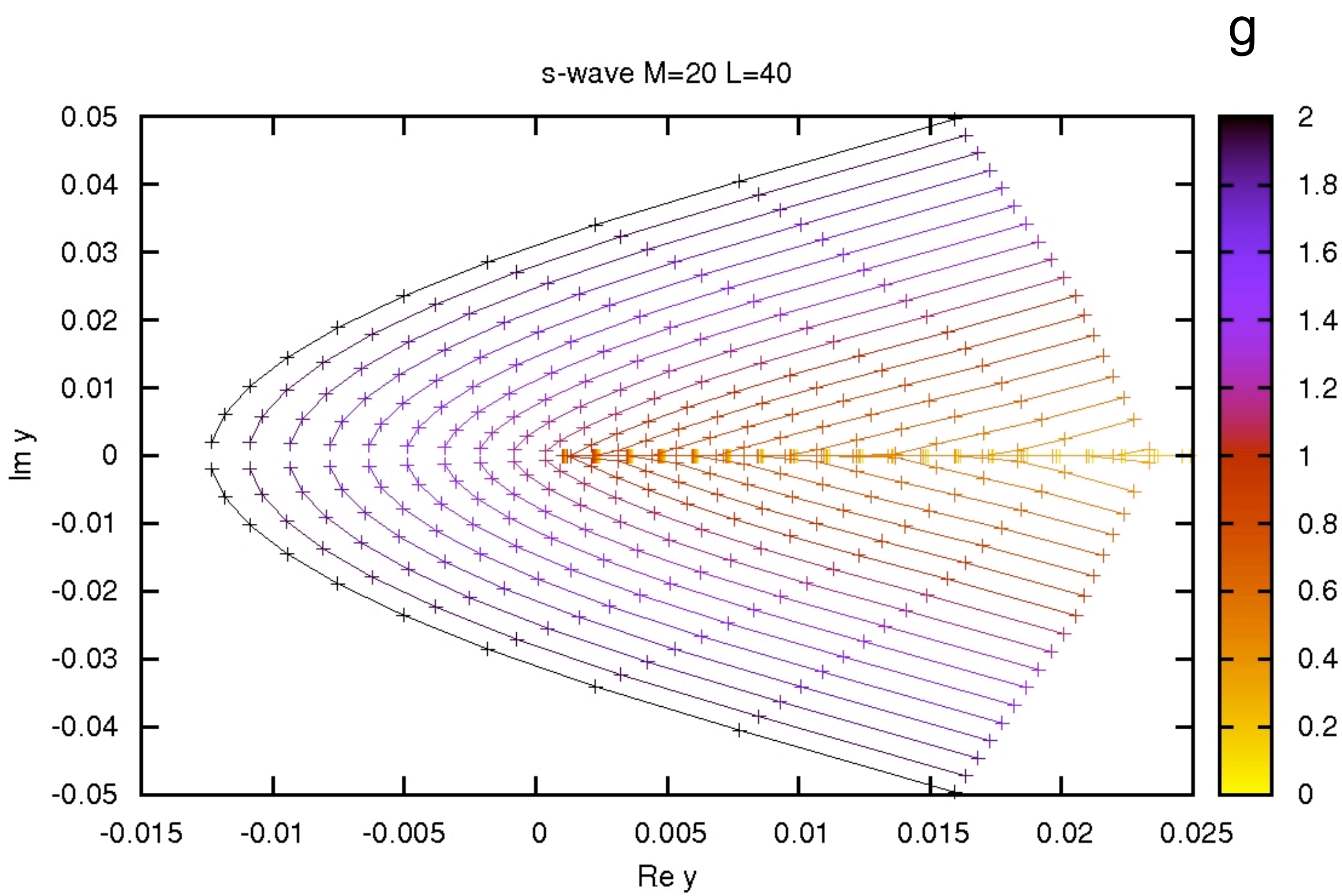
Re y_m



Roots in the complex y-plane (p + i p model)



Roots for the exactly solvable s-wave Richardson model



Electrostatic analogy of the BAEs

$$L, N \to \infty, \quad G \to 0, \quad with \quad x = \frac{N}{L}, \quad g = GL \quad finite$$

Let us assume that the roots y_m form an arc Γ in the complex plane with a density $r(y)$

The energies $\varepsilon = z_k^2$ form another arc Ω with density $\rho(\varepsilon)$

The BAEs become

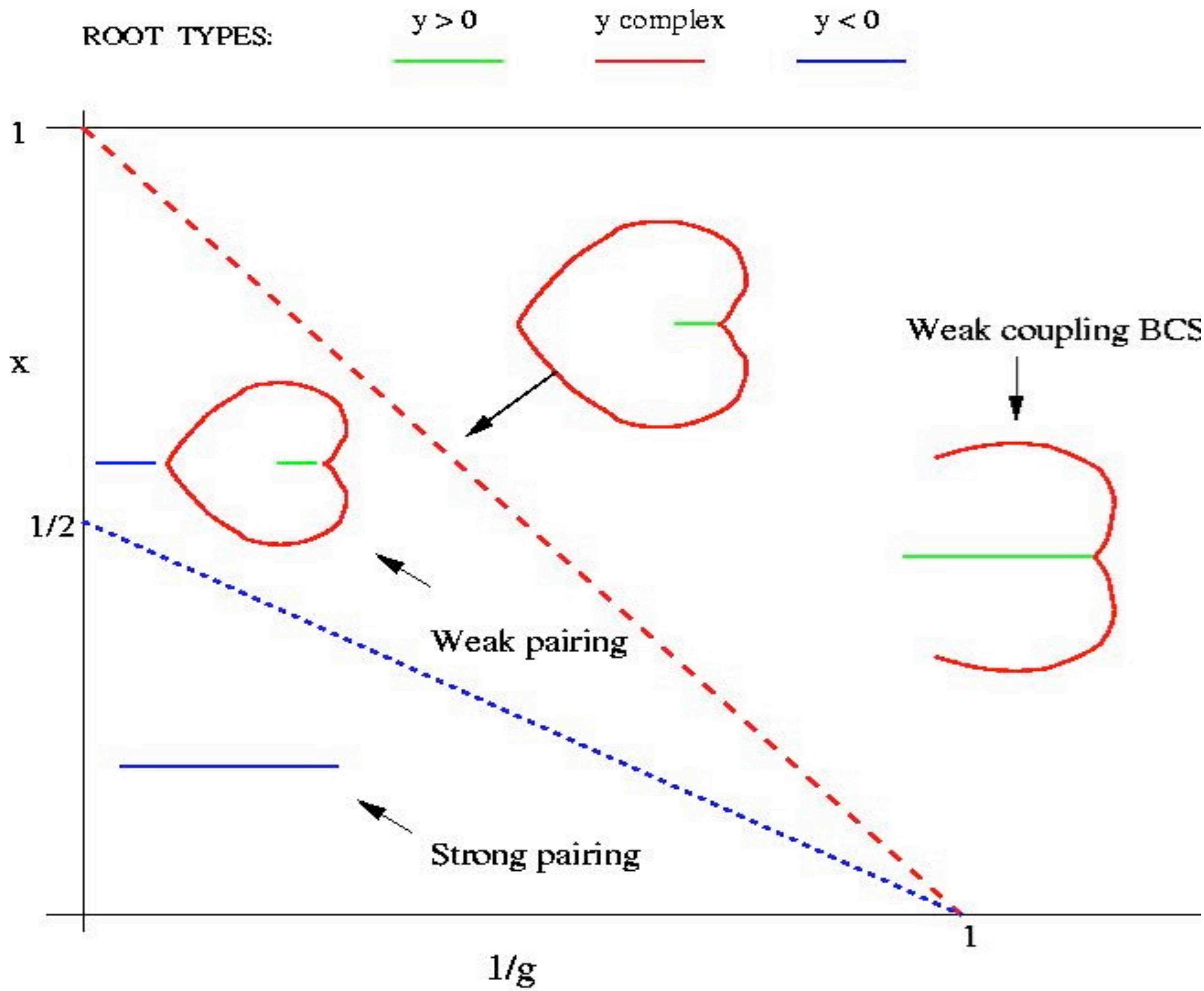
$$\int_\Omega d\varepsilon \frac{\rho(\varepsilon)}{\varepsilon - y} - \frac{q_0}{y} - P\int_\Gamma |dy'| \frac{r(y')}{y' - y} = 0, \quad y \in \Gamma$$

$$q_0 = \frac{1}{2G} - \frac{L}{2} + N$$

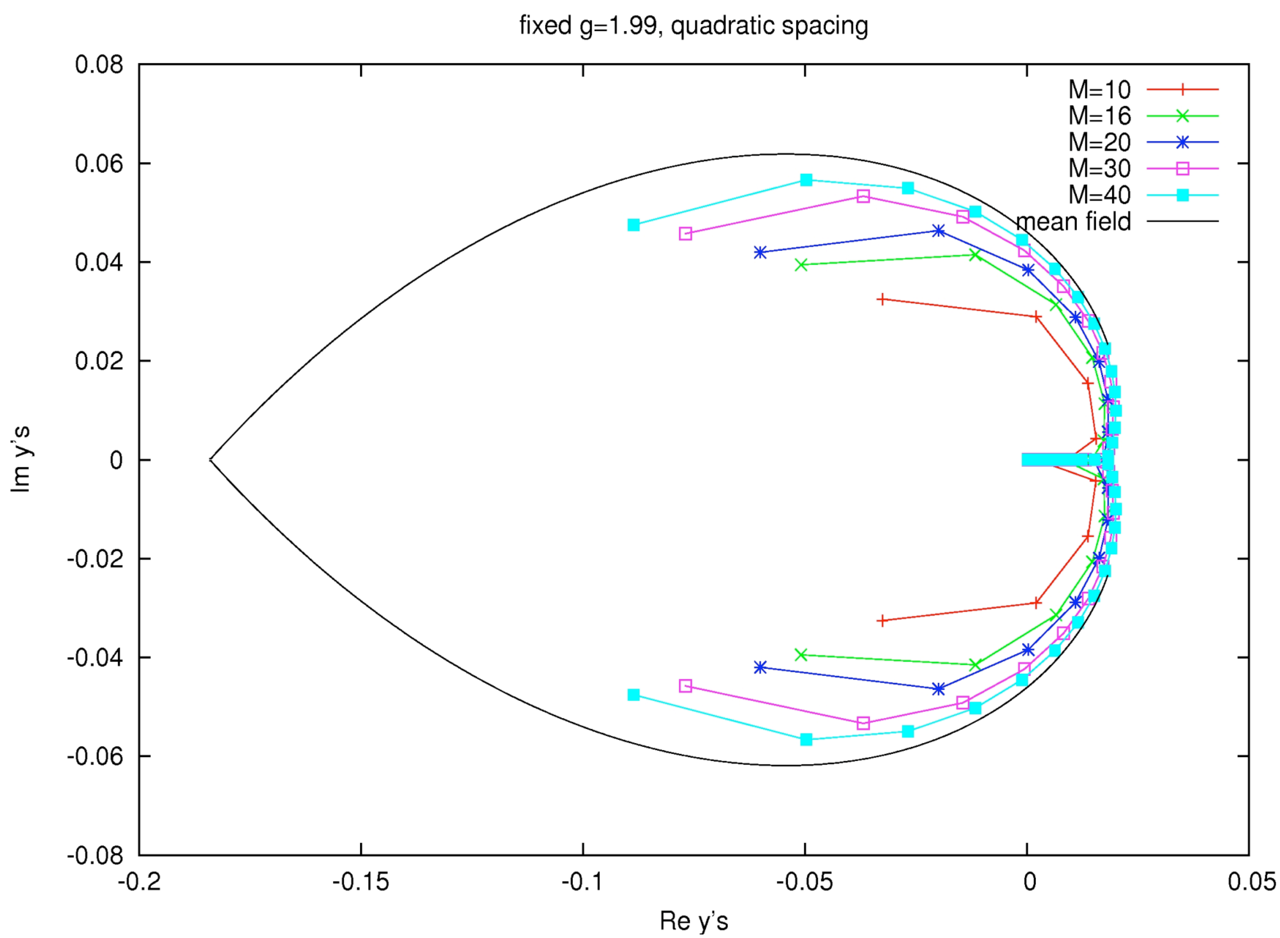
$$N = \int_\Gamma |dy| r(y), \quad E = \int_\Gamma |dy| y\, r(y)$$

There is a analytic solution of these equations which agree with the mean-field solution to leading order in L and N

Structure of the arcs formed by the roots y

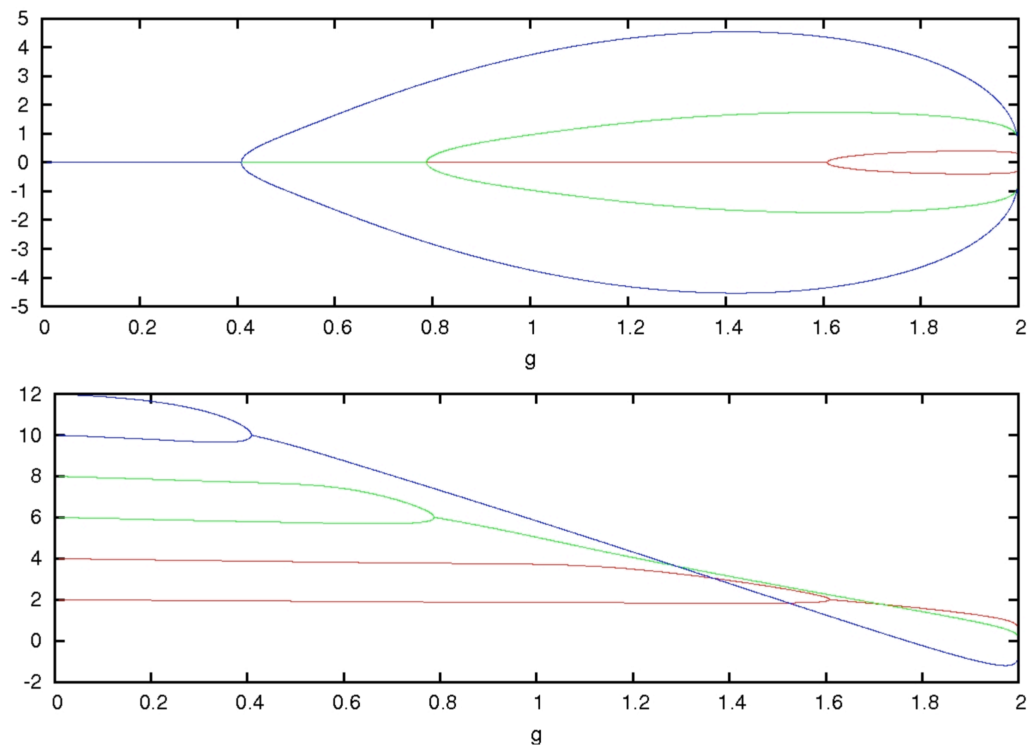


The equation of the complex arc can be determined and compared with the numerical results:
Example with x = 1/2 and g = 1.99 (weak coupling region)



What happens exactly at g = 2?

At x = 1/2 and g = 2 all the roots collapse to zero!!



More generally at

$$x = x_{MR} = 1 - \frac{1}{g} \quad \Rightarrow \quad y_m = 0 \quad \forall m$$

The Moore-Read line

Recall

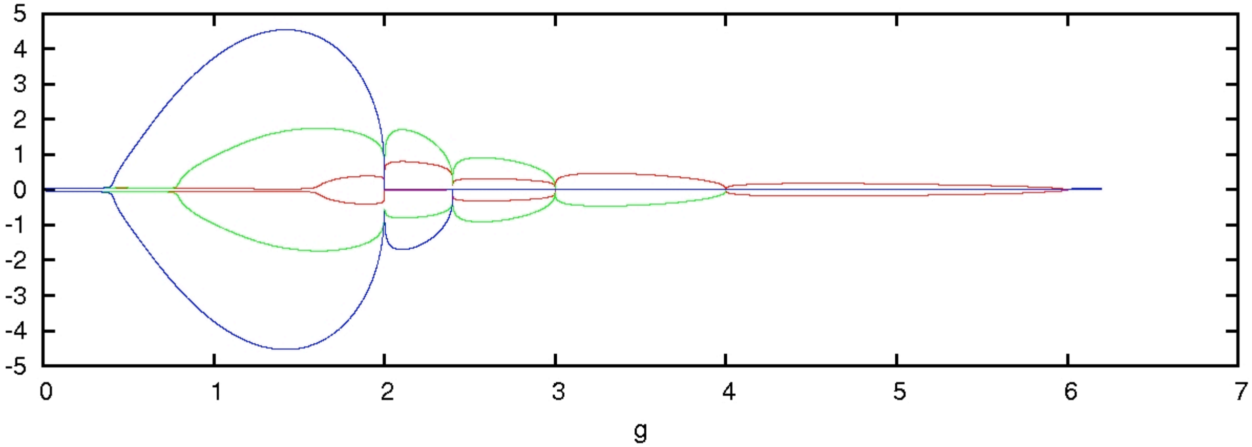
$$|\psi\rangle = \prod_{m=1}^{N} C(y_m)|0\rangle, \qquad C(y) = \sum_{k_x \geq 0, k_y} \frac{k_x - i k_y}{\vec{k}^2 - y} \; c^*_{\vec{k}} \; c^*_{-\vec{k}}$$

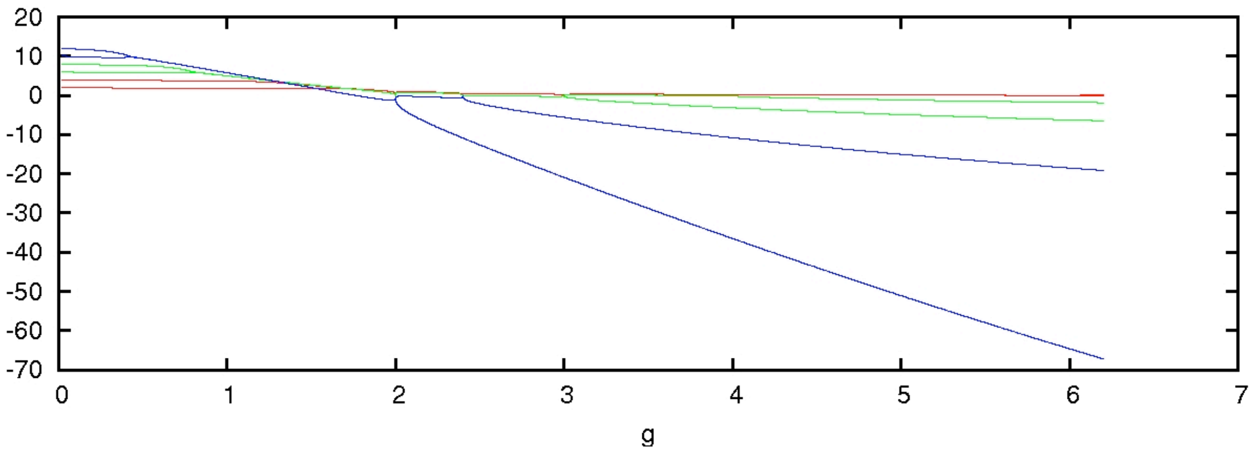
$$C(0) = \sum_{k_x \geq 0, k_y} \frac{1}{k_x + i k_y} \; c^*_{\vec{k}} \; c^*_{-\vec{k}} \;\; \rightarrow |\psi\rangle = [C(0)]^N |0\rangle,$$

Conclusion: The Moore-Read state is the exact ground state on the MR-line

Weak pairing region

At fixed values of g a some roots collapse to zero while others remaing real





What are the conditions for collapse?

N_W : number of roots in the weak pairing phase

$$N_W = N_0 + N_S$$

N_0 : number of zero roots (y=0)

N_S : number of non zero roots

The collapse of roots happens iff

$$\boxed{\frac{N_0}{L} + 2\frac{N_S}{L} = 1 - \frac{1}{g}}$$

Note.- g is a rational number

Moreover the N_S roots satisfy the BAEs in the strong pairing phase

Weak-strong pairing duality: dressing operation

Take a strong pairing eigenstate |S> and dress it with MR pairs:

$$DRESSING: \quad |W\rangle = [C(0)]^{N_0} |S\rangle$$

Then |W> is a weak pairing eigenstate at the same coupling g:

$$H|S\rangle = E|S\rangle \Rightarrow H|W\rangle = E|W\rangle$$

$$\frac{N_0 + N_S}{L} + \frac{N_S}{L} = 1 - \frac{1}{g} \Leftrightarrow \quad x_I + x_{II} = 1 - \frac{1}{g}$$

Weak-strong duality

The weak pairing phase is a sort of “two fluid” state made of strong pairing pairs and MR pairs

Discontinuity of the GS energy on the MR line

Take one pair for g >1. Its energy in the L>> 1 limit is finite

$$\frac{|y_1|}{\omega}\log\left(1+\frac{\omega}{|y_1|}\right)=1-\frac{1}{g}$$

Dress this pair with N_0 MR pairs

$$x_I + x_{II} = x_I + \frac{1}{L} = 1-\frac{1}{g} \rightarrow x_I = 1-\frac{1}{g}-\frac{1}{L} \approx 1-\frac{1}{g} = x_{MR}$$

$$\lim_{x\rightarrow x_{MR}} E_0(g,x) = |y_1| \neq E_0(g,x_{MR}) = 0$$

One may call this a zeroth order phase transition

Questions and suggestions

-The Moore-Read line is a crossover or a phase transition line?
If so, what sort of phase transition: topological?

- There is a gap in the spectrum except on the Read-Green line. Its value agrees with mean field theory.

- Is it well defined the thermodynamic limit in the weak pairing phase?

- We can compute scalar products and expectations values of some operators. So quenching studies can be performed as those done by Caux and Calebrese for the s-wave model.

-Is there any signature of non abelian anyons?
Need to go beyond the present model.

Thank you

Obrigado