Workshop on Correlations and Coherence in Quantum Matter

Fidelity susceptibility and quantum phase transitions

Speaker: Shi-Jian Gu

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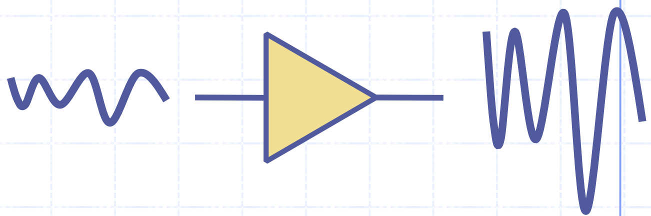

$$F(A,B) = \langle \Psi(A) | \Psi(B) \rangle$$

Evora, Portugal, 10-14 November 2008

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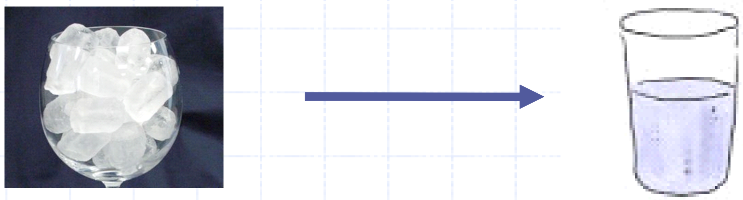
Content

Introduction: QPT

Thermal phase transitions: which is described by non-analytic behaviors of the thermal properties at the transition points, driven by thermal fluctuation.

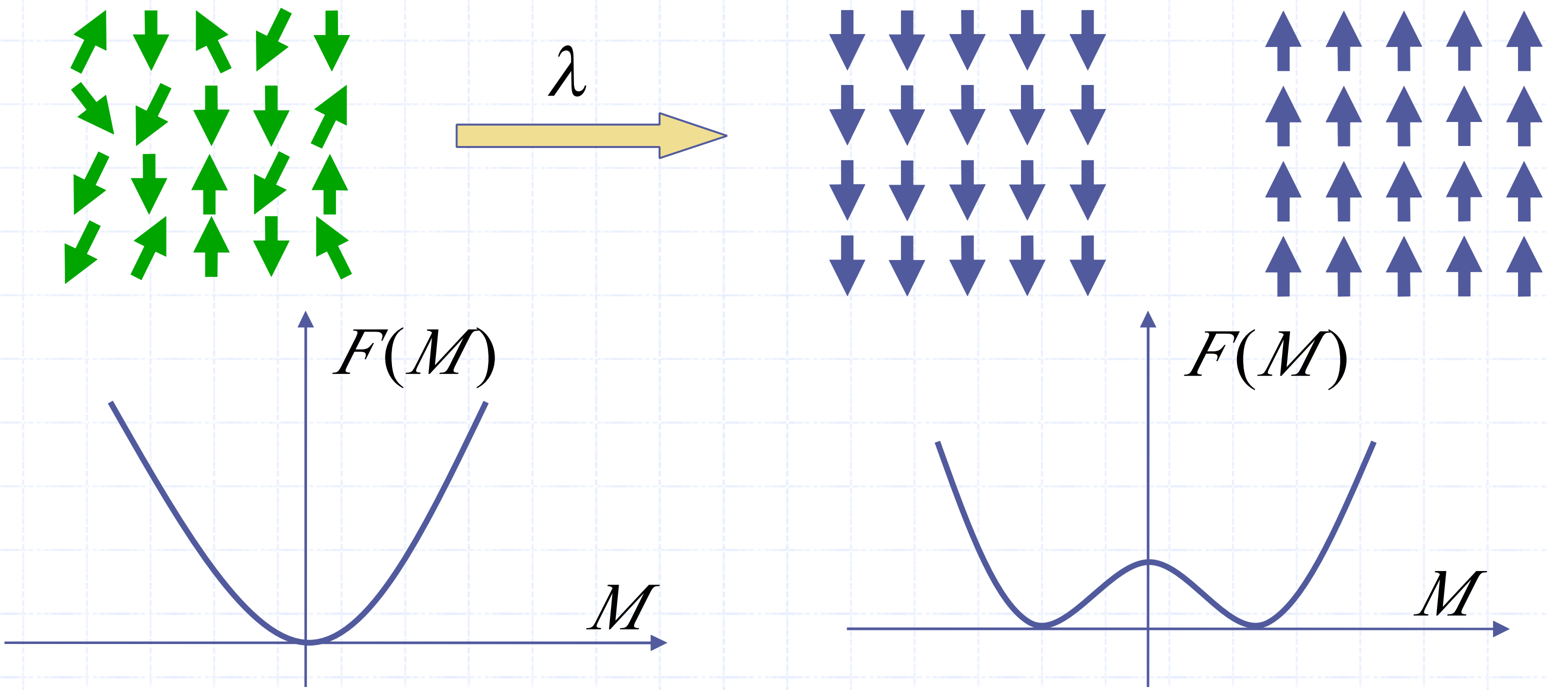


Quantum phase transitions: driven by the quantum fluctuations and are described by the non-analytic behaviors of the ground-state properties at the transition points.

- High Tc superconductor
- Mott-insulator transition in Hubbard model.

Introduction: traditional method

Landau's symmetry-breaking theory



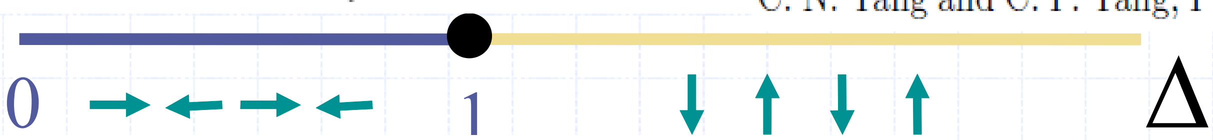
$$F(M) = AM^2 + BM^4 +$$
$$A > 0, B > 0$$

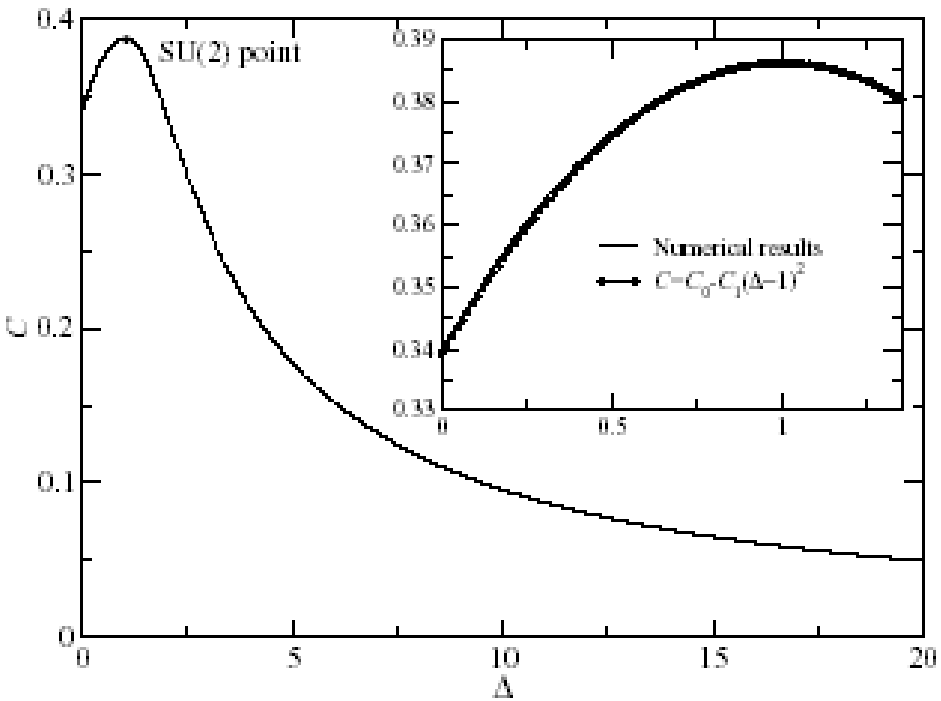
$$F(M) = AM^2 + BM^4 +$$
$$A < 0, B > 0$$

Introduction: QPT & quantum entanglement

$$\hat{H}_{\rm XXZ} = \sum_i \left(\hat{s}_i^x \hat{s}_{i+1}^x + \hat{s}_i^y \hat{s}_{i+1}^y + \Delta \hat{s}_i^z \hat{s}_{i+1}^z \right) ;$$

C. N. Yang and C. P. Yang, Phys. Rev. **150**, 321 (1966)





$$C_{ij} \approx \left\langle s_i^x s_j^x \right\rangle + \left\langle s_i^x s_j^x \right\rangle + \left\langle s_i^x s_j^x \right\rangle$$

$$C = C_0 - C_1 (\Delta - 1)^2$$

$$C_0 = 2\ln 2 - 1$$

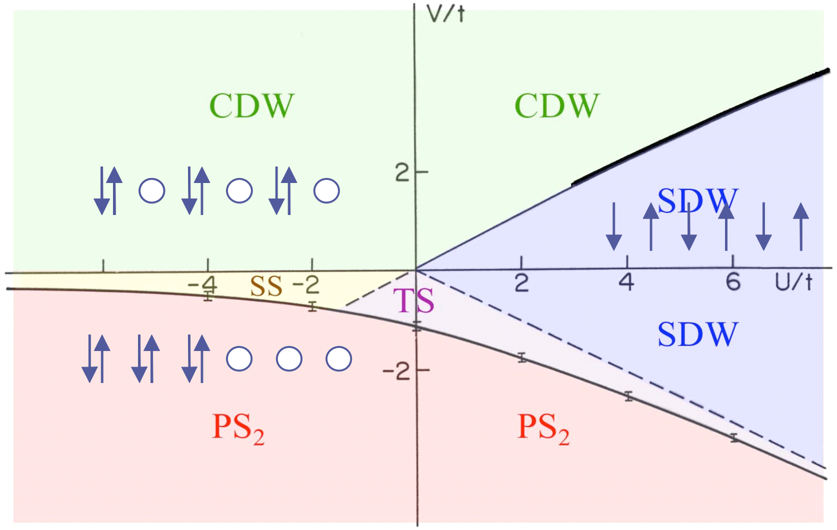
$$C_1 = 2\ln 2 - \frac{1}{2} - \frac{2}{\pi} - \frac{2}{\pi^2}$$

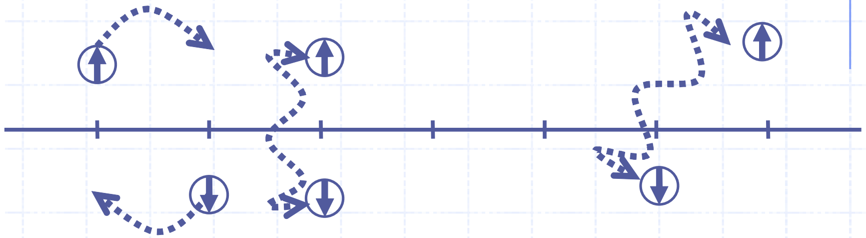
Gu etal, PRA, 68, 042330(2003).

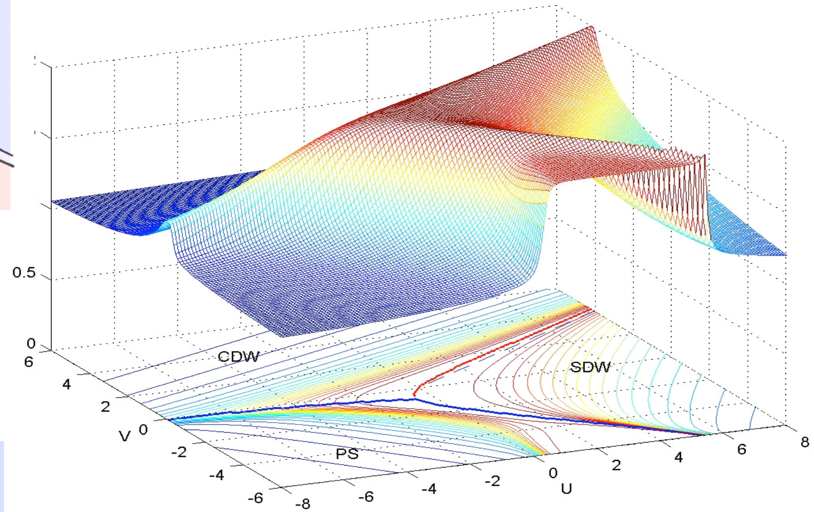
Introduction: QPT & quantum entanglement

The extended Hubbard model

$$H = -\sum_{\sigma,j,\delta} c^{+}_{j,\sigma} c_{j+\delta,\sigma} + U\sum_{j} n_{j,\uparrow} n_{j,\downarrow} + V\sum_{j} n_{j} n_{j+1}$$







Gu, etal, PRL. 93, 086402 (2004).

Introduction: QPT & quantum entanglement

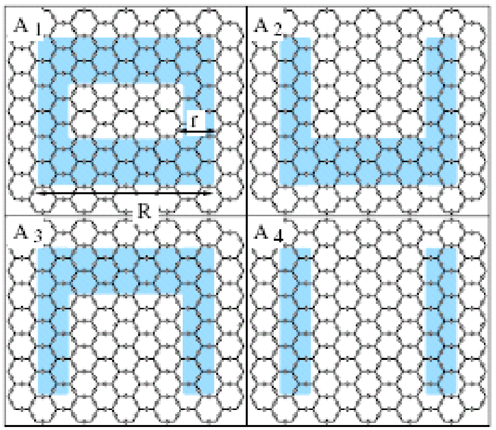
Detecting Topological Order in a Ground State Wave Function Kitaev&Preskill

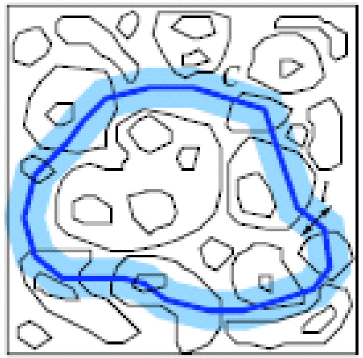
Michael Levin and Xiao-Gang Wen

Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

(Received 25 October 2005; published 24 March 2006)

A large class of topological orders can be understood and classified using the string-net condensation picture. These topological orders can be characterized by a set of data $(N, d_i, F^{ijk}_{lmn}, \delta_{ijk})$. We describe a way to detect this kind of topological order using only the ground state wave function. The method involves computing a quantity called the "topological entropy" which directly measures the total quantum dimension $D = \sum_i d_i^2$.

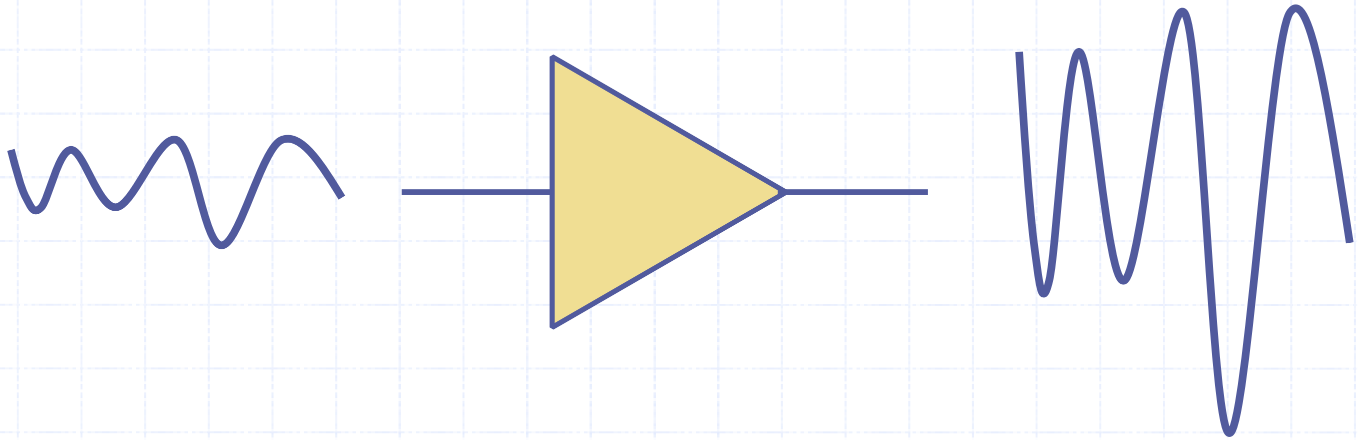




$$(S_1 - S_2) - (S_3 - S_4)$$

1D: G. Vidal, J. I. Latorre, E. Rico, and A. Kitaev, Phys. Rev. Lett. **90**, 227902 (2003).

Introduction: classical fidelity



Definition

$$\rho = p_1|1\rangle\langle 1| + p_2|2\rangle\langle 2| + \cdots + p_N|N\rangle\langle N|$$

$$\sigma = q_1|1\rangle\langle 1| + q_2|2\rangle\langle 2| + \cdots + q_N|N\rangle\langle N|$$

$$F = \sum_i \sqrt{p_i q_i}$$

Introduction: quantum fidelity

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R. Jozsa, J. Mod. Opt. 41, 2315 (1994).

$$F(\Psi', \Psi) = |\langle \Psi' | \Psi \rangle| \qquad \mathbf{a} \cdot \mathbf{b} = ab\cos(\theta)$$

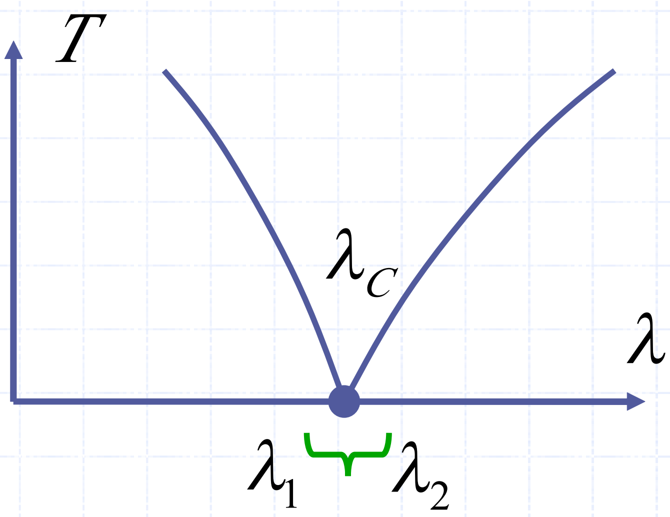
$$\begin{aligned} |\Psi(\theta)\rangle &= \cos\theta|\uparrow\downarrow\rangle + \sin\theta|\downarrow\uparrow\rangle, \\ |\Psi(\theta')\rangle &= \cos\theta'|\uparrow\downarrow\rangle + \sin\theta'|\downarrow\uparrow\rangle, \end{aligned}$$

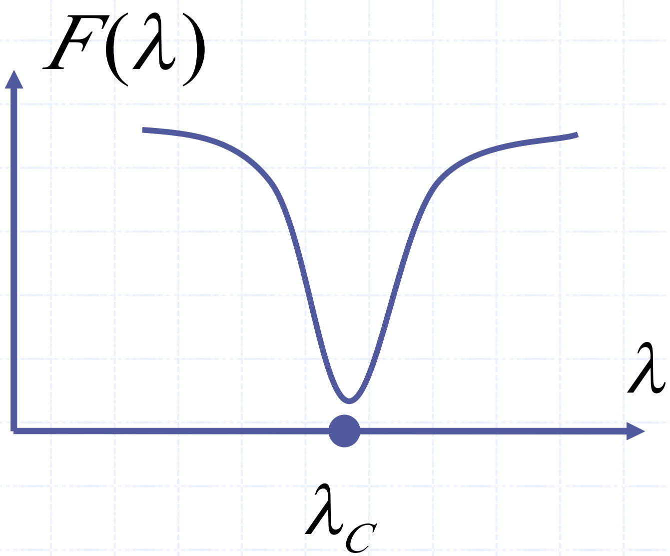
$$F(\Psi(\theta'), \Psi(\theta)) = |\cos(\theta - \theta')|.$$

$$F(\rho, \sigma) = \sqrt{\rho^{1/2}\sigma\rho^{1/2}}$$

$$\rho = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad \sigma = \begin{pmatrix} c & \\ & d \end{pmatrix} \qquad F(\rho, \sigma) = \sqrt{ac} + \sqrt{bd}$$

Introduction: information perspective





$\psi(\lambda_1) \qquad \psi(\lambda_2)$

$$F = |\langle \psi(\lambda_1) | \psi(\lambda_2) \rangle|$$

Introduction: QPT & Fidelity

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Introduction: QPT & Fidelity

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Introduction: Loschmidt echo in many-body system

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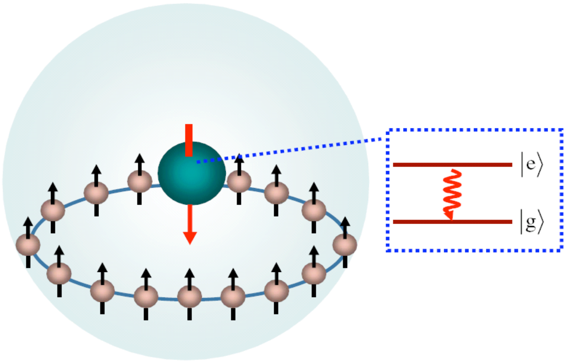
Introduction: QPT & Fidelity

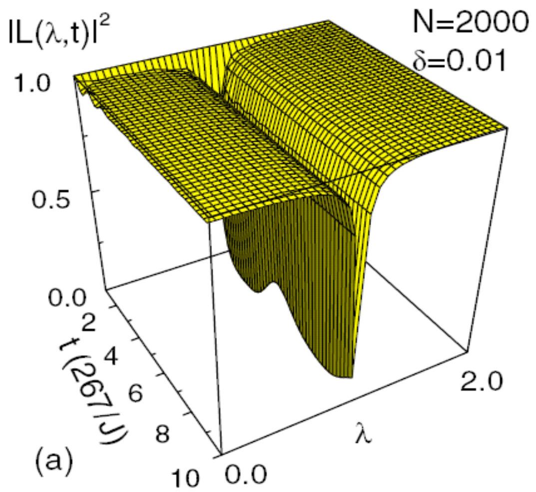
Ising model

H. T. Quan, Z. Song, X. F. Liu, P. Zanardi, and C. P. Sun, Phys. Rev. Lett. **96**, 140604 (2006).

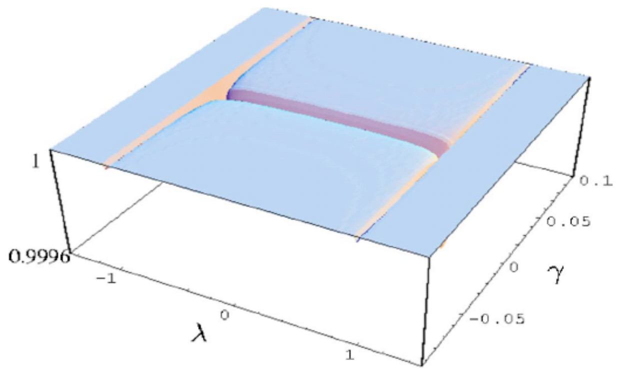
$$H(\lambda, \delta) = -J\sum_j(\sigma_j^z\sigma_{j+1}^z + \lambda\sigma_j^x + \delta|e\rangle\langle e|\sigma_j^x),$$



$$L(\lambda, t) = |\langle\varphi_g(t)|\varphi_e(t)\rangle|^2.$$



$$\hat{H}(\gamma,\lambda) = -\sum_{i=-M}^{M}\left(\frac{1+\gamma}{2}\hat{\sigma}_i^x\hat{\sigma}_{i+1}^x + \frac{1-\gamma}{2}\hat{\sigma}_i^y\hat{\sigma}_{i+1}^y + \frac{\lambda}{2}\hat{\sigma}_i^z\right).$$

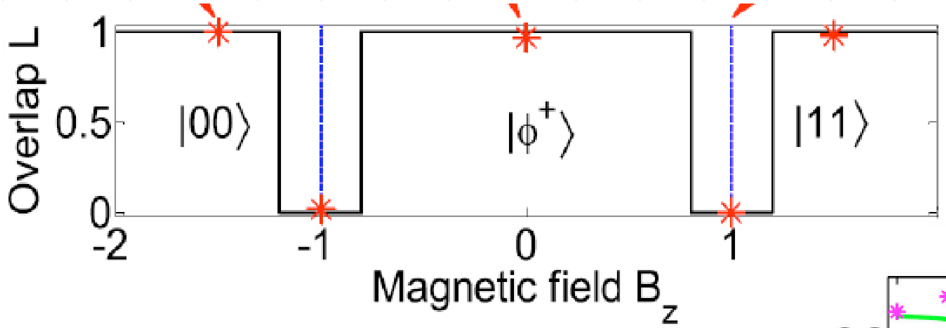


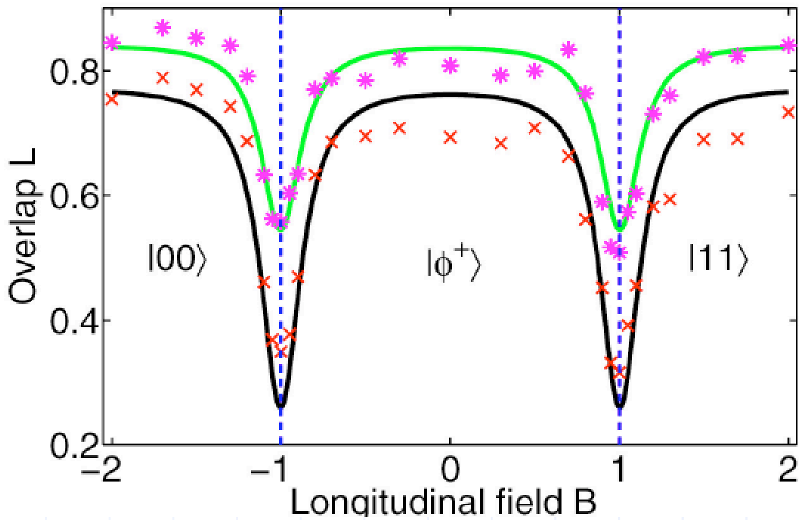
P. Zanardi and N. Paunkovic, Phys. Rev. E **74**, 031123 (2006).

Introduction: QPT & Fidelity

J. Zhang, X. Peng, N. Rajendran, and D. Suter, Phys. Rev. Lett. **100**, 100501 (2008).

$$H^s = \sigma_1^z\sigma_2^z + B_x(\sigma_1^x + \sigma_2^x) + B_z(\sigma_1^z + \sigma_2^z)$$





Relevant works to the talks (published):

1. W. L. You, Y. W. Li, and **S. J. Gu**, Phys. Rev. E **76**, 022101 (2007).
2. S. Chen, L. Wang, **S. J. Gu**, and Y. P. Wang, Phys. Rev. E **76**, 061108 (2007).
3. **S. J. Gu**, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B 77, 245109 (2008)
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5. W. Q. Ning, **S. J. Gu**, C. Q. Wu, and H. Q. Lin, J. Phys.: Condens. Matter 20 235236 (2008).
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1. Dimension of fidelity susceptibility in quantum phase transitions, **Shi-Jian Gu** and Hai-Qing Lin, submitted
2. Scaling of reduced fidelity susceptibility in the one-dimensional transverse-field XY model
   Wen-Long You,Wen-Long Lu, Xiaoguang Wang, and **Shi-Jian Gu**, submitted.
3. Density-functional fidelity approach to quantum phase transitions, **Shi-Jian Gu**, submitted.
4. Fidelity susceptibility for SU(2)-invariant states
   Xiaoguang Wang, **Shi-Jian Gu**

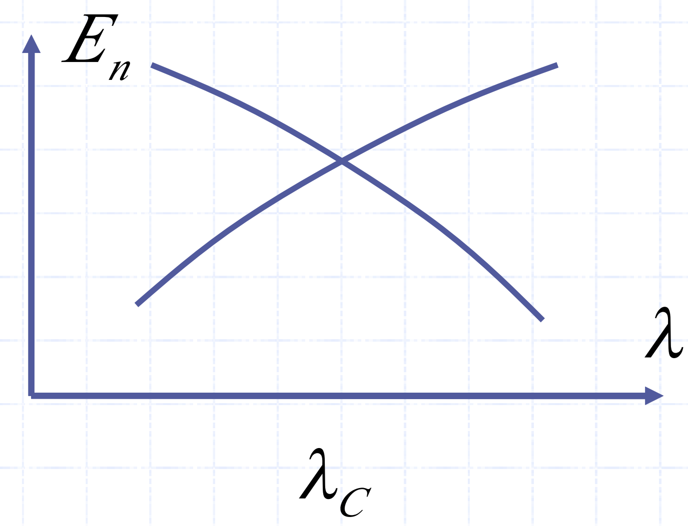
Content

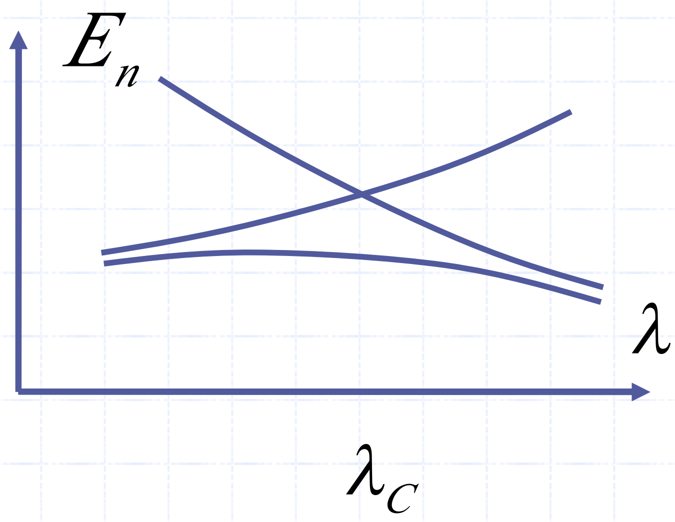
Spectra reconstruction

How does QPT happen for a general quantum system

$$H(\lambda) = H_0 + \lambda H_I,$$

$$H(\lambda)|\Psi_n(\lambda)\rangle \;=\; E_n|\Psi_n(\lambda)\rangle$$





Perturbation method in quantum mechanics

Fidelity susceptibility

$$|\Psi_0(\lambda+\delta\lambda)\rangle = |\Psi_0(\lambda)\rangle + \delta\lambda \sum_{n\neq 0} \frac{H_{n0}(\lambda)|\Psi_n(\lambda)\rangle}{E_0(\lambda) - E_n(\lambda)}$$

$$H_{n0} = \langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle.$$

$$F_i(\lambda,\delta) = |\langle \Psi_0(\lambda)|\Psi_0(\lambda+\delta)\rangle|$$

$$\frac{1}{F_i^2} = 1 + \delta\lambda^2 \sum_{n\neq 0} \frac{|\langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2}$$

$$\chi_F \equiv \lim_{\delta\lambda\to 0} \frac{-2\ln F_i}{\delta\lambda^2}$$

You, Li, and Gu, PRE, 76, 022101 (2007)

Zanardi, Giorda, and Cozzini, PRL99, 100603 (2007).

$$\chi_F(\lambda) = \sum_{n\neq 0} \frac{|\langle \Psi_n(\lambda)|H_I|\Psi_0(\lambda)\rangle|^2}{[E_n(\lambda) - E_0(\lambda)]^2}$$

Perturbation method in quantum mechanics

Fidelity susceptibility: what is the physics

$$\chi_F(\omega) = \sum_{n\neq 0} \frac{|\langle\Psi_n|H_I|\Psi_0\rangle|^2}{[E_n - E_0]^2 + \omega^2}$$

You, Li, and Gu, PRE, 76, 022101 (2007)

$$\chi_F = \int \tau \left[\langle\Psi_0|H_I(\tau)H_I(0)|\Psi_0\rangle - \langle\Psi_0|H_I|\Psi_0\rangle^2\right] d\tau$$

Fidelity susceptibility <==> dynamic structure factor

$$[H_0, H_I] = 0 \longrightarrow \chi_F = 0$$

Extension to thermal phase transitions

Fidelity susceptibility: extension to TPT

P. Zanardi, H. T. Quan, X. Wang, and C. P. Sun, Phys. Rev. A **75**, 032109 (2007).

$$F_i(\beta, \delta) = \frac{Z(\beta)}{\sqrt{Z(\beta - \delta\beta/2)Z(\beta + \delta\beta/2)}},$$

You, Li, and Gu, PRE, 76, 022101 (2007)

Zanardi, Giorda, and Cozzini, PRL99, 100603 (2007).

$$\chi_F = \left.\frac{-2\ln F_i}{\delta\beta^2}\right|_{\delta\beta\to 0} = \frac{C_v}{4\beta^2} \qquad C_v = \beta^2(\langle E^2\rangle - \langle E\rangle^2)$$

$$\chi_F = \left.\frac{-2\ln F_i}{\delta h^2}\right|_{\delta h\to 0} = \frac{\beta\chi}{4} \qquad \chi = \beta(\langle \dot{M}^2\rangle - \langle \dot{M}\rangle^2)$$

Content

Universality class described by the FS

$$H_I = \sum_r V(r)$$

L. C. Venuti and P. Zanardi, Phys. Rev. Lett. **99** , 095701 (2007).

$$\chi_F = \int \tau \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] d\tau$$

$$C(r,\tau) = \langle \Psi_0(\lambda) | V(r,\tau) V(0,0) | \Psi_0(\lambda) \rangle$$
$$- \langle \Psi_0(\lambda) | V(r,0) | \Psi_0(\lambda) \rangle \langle \Psi_0(\lambda) | V(0,0) | \Psi_0(\lambda) \rangle$$

$$r' = s\, r, \quad \tau' = s^{\zeta} \tau, \quad V(r') = s^{-\Delta_V} V(r)$$

$$\frac{\chi_F}{L^d} \sim L^{d+2\zeta-2\Delta_V}$$

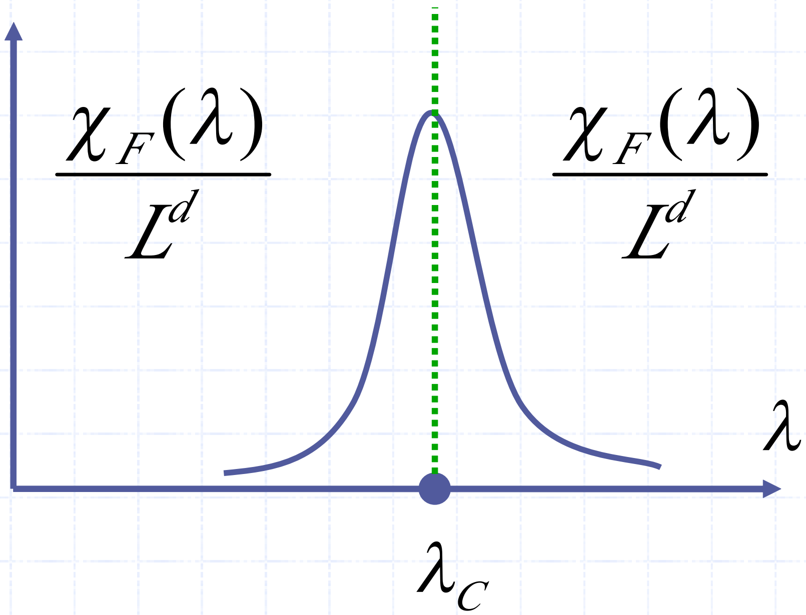
S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B **77**, 245109 (2008).

$$\frac{\chi_F}{L^d} \sim \frac{1}{|\lambda - \lambda_c|^{\alpha}} \qquad \alpha = \frac{d + 2\zeta - 2\Delta_V}{\nu}$$

Adiabatic dimension of fidelity susceptibility

- The fidelity susceptibility

$$\frac{\chi_F}{L^d} \sim \frac{1}{|\lambda - \lambda_c|^\alpha} \qquad \alpha = \frac{d + 2\zeta - 2\Delta_V}{\nu}$$



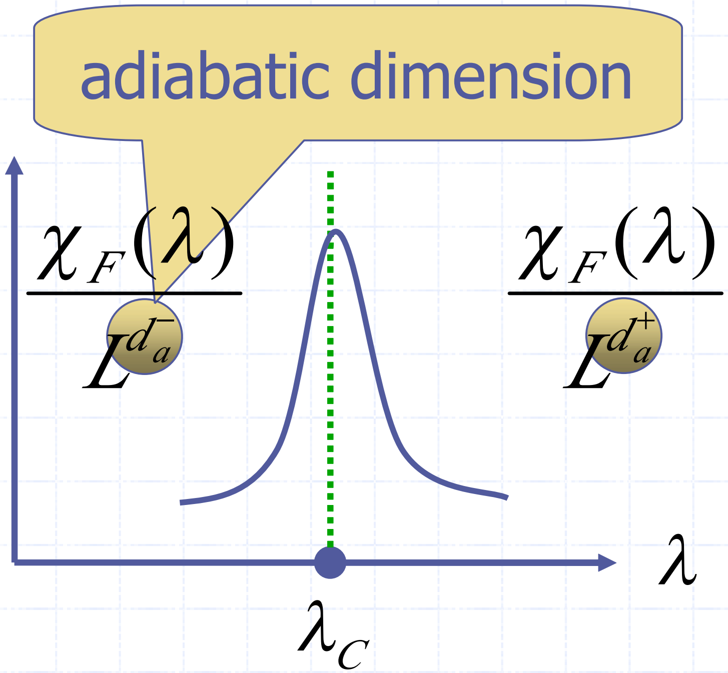
Adiabatic dimension of fidelity susceptibility

In general quantum phases

Gu and Lin, arXiv: 08073491

$$C(r,\tau) = \frac{1}{r^{2\Delta_V}} f(r\tau^{1/\zeta})$$

$$\begin{aligned}\frac{\chi_F}{L^d} &= \sum_r \int \frac{\tau}{r^{2\Delta_V}} f(r\tau^{1/\zeta}) d\tau, \\ &\sim \sum_r \frac{1}{r^{2\Delta_V - 2\zeta}} \\ &\propto \begin{cases} L^{d+2\zeta-2\Delta_V}, & 2\Delta_V - 2\zeta \neq d \\ \ln L, & 2\Delta_V - 2\zeta = d \end{cases}\end{aligned}$$

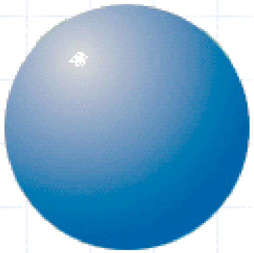


$$\frac{\chi_F}{L^{d_a^\pm}} \sim \frac{1}{|\lambda - \lambda_c|^{\alpha^\pm}} \qquad \frac{\chi(\lambda, L)}{L^{d_a^\pm}} = \frac{A}{L^{-\mu + d_a^\pm} + B(\lambda - \lambda_{\max})^{\alpha^\pm}}$$

Quantum and classical distance (unpublished)

$$f\Delta t = m(v_2 - v_1)$$

$$f\Delta x = \tfrac{1}{2} m(v_2^2 - v_1^2)$$



$$F\Delta t = M(v_2 - v_1)$$

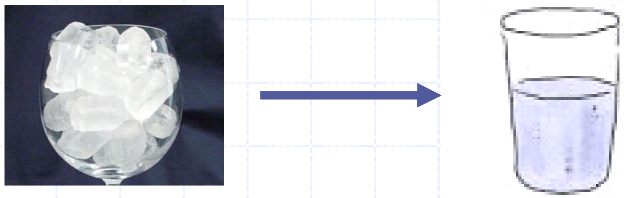
$$F\Delta x = \tfrac{1}{2} M(v_2^2 - v_1^2)$$

$$F = ma$$

Quantum and classical distance (unpublished)

$$F_i(\beta,\delta) = \frac{Z(\beta)}{\sqrt{Z(\beta-\delta\beta/2)Z(\beta+\delta\beta/2)}},$$

$$\chi_F = \left.\frac{-2\ln F_i}{\delta\beta^2}\right|_{\delta\beta\to 0} = \frac{C_v}{4\beta^2}$$



Logarithmic fidelity

$$\ln F \approx L^d$$

$$F = |\langle\psi(\lambda_1)|\psi(\lambda_2)\rangle|$$

$$\frac{\chi_F}{L^d} = \sum_r \int \tau G(r,\tau)d\tau$$

$$\ln F \approx L^{d_a}$$

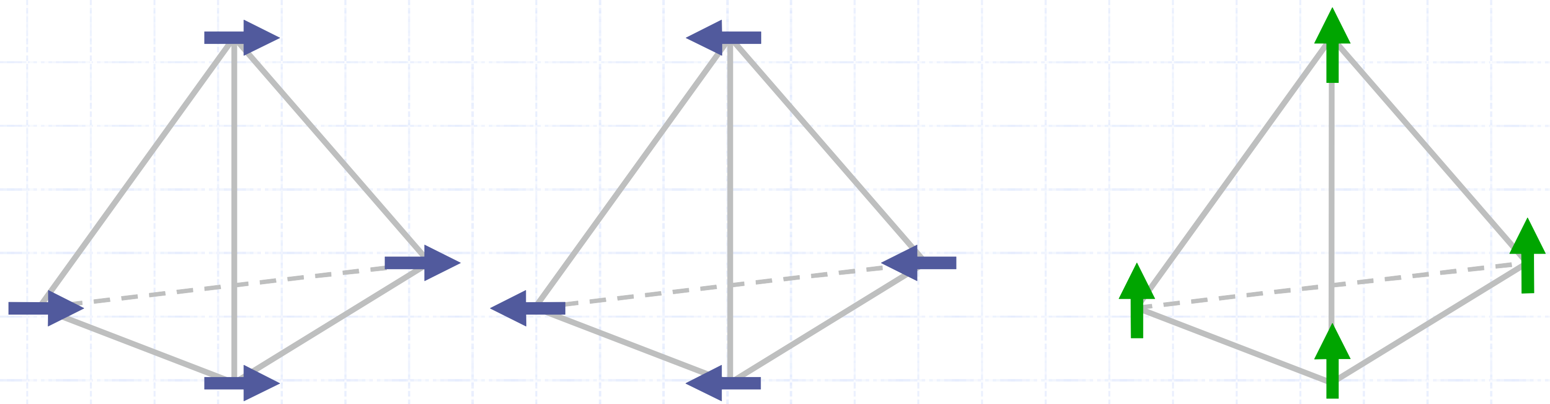
Application: the Lipkin-Meshkov-Glick model

Hamiltonian

$$H = -\frac{\lambda}{N}\sum_{i<j}\left(\sigma_x^i\sigma_x^j + \gamma\sigma_y^i\sigma_y^j\right) - h\sum_i \sigma_z^i,$$

Ground phases (ferromagnetic)

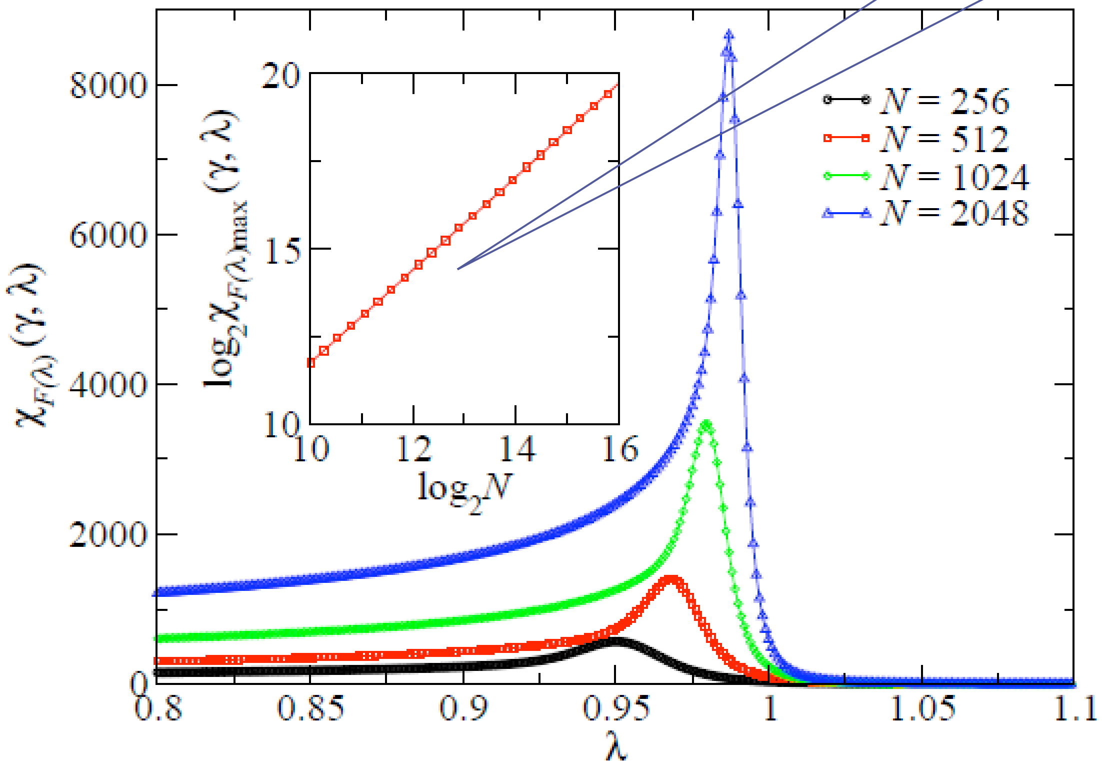
$h<1$ $\qquad\qquad\qquad\qquad$ $h>1$



Application: the LMG model

$\gamma = 0.5$

$\mu \simeq 1.33$



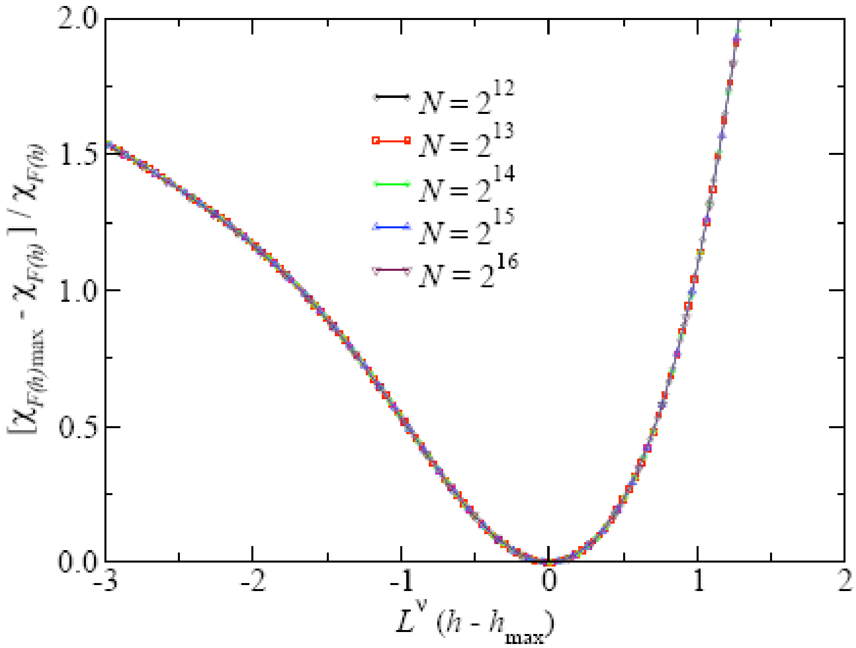
$\chi_F \propto N^0, \quad d_a^+ = 0$

$\chi_F \propto N, \quad d_a^- = 1$

Application: the LMG model

$$\frac{\chi_{F(h)}(h_{\max},\eta)-\chi_{F(h)}(h,\eta)}{\chi_{F(h)}(h,\eta)}=f[L^{\nu}(h-h_{\max})]$$

$$\mu \simeq 1.33$$



$$\frac{\chi_F}{L^{d_a^{\pm}}}\sim\frac{1}{|\lambda-\lambda_c|^{\alpha^{\pm}}}$$

$$\alpha^{\pm}=\frac{\mu-d_a^{\pm}}{\nu}$$

$$\alpha^{\pm}=\begin{cases}1/2 & \lambda<1\\ 2 & \lambda>1\end{cases}$$

$$\nu \simeq 0.665.$$

Application: the LMG model

J. I. Latorre, R. Orús, E. Rico, and J. Vidal, Phys. Rev. A **71**, 064101 (2005).

If $h > 1$

$$S_z = S - a^\dagger a,$$
$$S_+ = (2S - a^\dagger a)^{1/2} a$$

The Hamiltonian in terms of bosons

$$H = -hN + [2(h-1) + \eta] a^\dagger a - \frac{\eta}{2}\left(a^{\dagger 2} + a^2\right)$$

$$a^\dagger = \cosh(\Theta/2) b^\dagger + \sinh(\Theta/2) b,$$
$$a = \sinh(\Theta/2) b^\dagger + \cosh(\Theta/2) b,$$

The diagonalized form

$$H = -h(N+1) + 2\sqrt{(h-1)(h-1+\eta)}\left(b^\dagger b + \frac{1}{2}\right)$$

Application: the LMG model

H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, Phys. Rev. E **78**, 032103 (2008).

If $h > 1$

$$\chi_{F(h)}(\eta, h > 1) = \frac{\eta^2}{32(h-1)^2(h-1+\eta)^2}$$

If $h < 1$

$$\frac{\chi_{F(h)}(\eta, h < 1)}{N} = \frac{1}{4\sqrt{(1-h^2)\eta}}$$

Exponents

$$\alpha = \begin{cases} 2, h > 1 \\ \frac{1}{2}, 0 \leq h < 1 \end{cases}$$

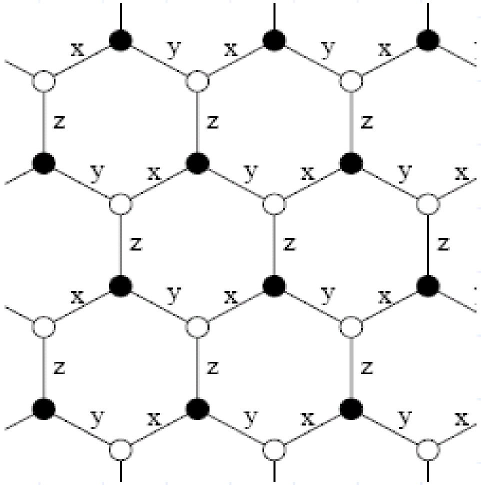
Content

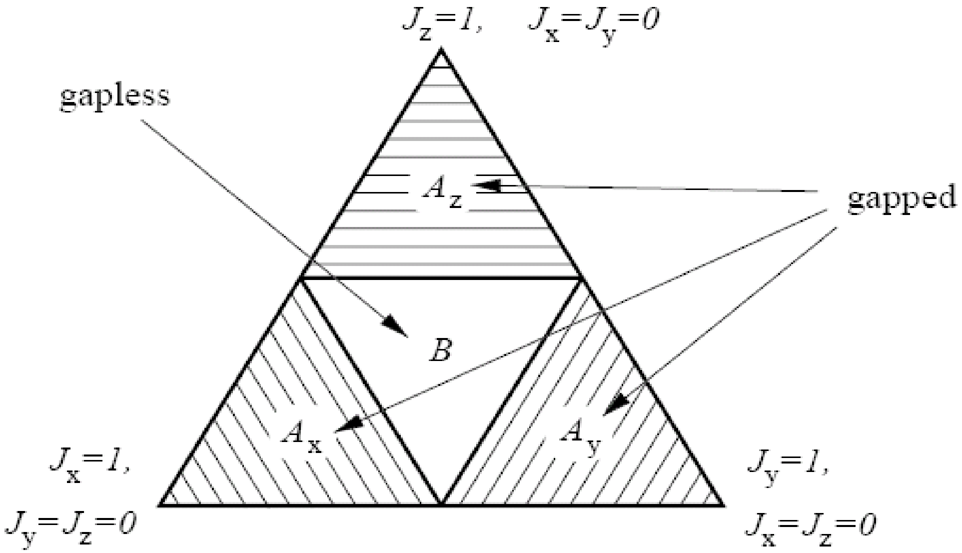
The Kitaev honeycomb model

A. Kitaev, Ann. Phys. **303**, 2 (2003); Ann. Phys. **321**, 2 (2006).

$$H = -J_x \sum_{x\text{-bonds}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-bonds}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-bonds}} \sigma_j^z \sigma_k^z,$$

$$J_x + J_y + J_z = 1$$





X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University, New York, 2004).

The critical fidelity susceptibility

$$\sigma^x = \mathrm{i}b^x c,\ \sigma^y = \mathrm{i}b^y c,\ \sigma^z = \mathrm{i}b^z c.$$

$$H = \frac{\mathrm{i}}{2}\sum_{j,k} \widehat{u}_{jk} J_{a_{jk}} c_j c_k$$

$$H = \sum_{\mathbf{q}} \begin{pmatrix} a_{-\mathbf{q},1} \\ a_{-\mathbf{q},2} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & \mathrm{i}f(\mathbf{q}) \\ -\mathrm{i}f(\mathbf{q})^* & 0 \end{pmatrix} \begin{pmatrix} a_{\mathbf{q},1} \\ a_{\mathbf{q},2} \end{pmatrix}$$

$$H = \sum_{\mathbf{q}} \sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2} \left(C_{\mathbf{q},1}^\dagger C_{\mathbf{q},1} - C_{\mathbf{q},2}^\dagger C_{\mathbf{q},2} \right)$$

$$\begin{aligned} f(\mathbf{q}) &= \epsilon_{\mathbf{q}} + \mathrm{i}\Delta_{\mathbf{q}}, \\ \epsilon_{\mathbf{q}} &= J_x \cos q_x + J_y \cos q_y + J_z \\ \Delta_{\mathbf{q}} &= J_x \sin q_x + J_y \sin q_y. \end{aligned}$$

A. Kitaev, Ann. Phys. **303**, 2 (2003); Ann. Phys. **321**, 2 (2006).

The critical fidelity susceptibility

$$
\begin{aligned}
|\Psi_0\rangle &= \prod_{\mathbf{q}} C^{\dagger}_{\mathbf{q},2}\,|0\rangle \\
&= \prod_{\mathbf{q}} \frac{1}{\sqrt{2}} \left(\frac{\sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}}{\Delta_{\mathbf{q}} + i\epsilon_{\mathbf{q}}} a_{-\mathbf{q},1} + a_{-\mathbf{q},2} \right) |0\rangle
\end{aligned}
$$

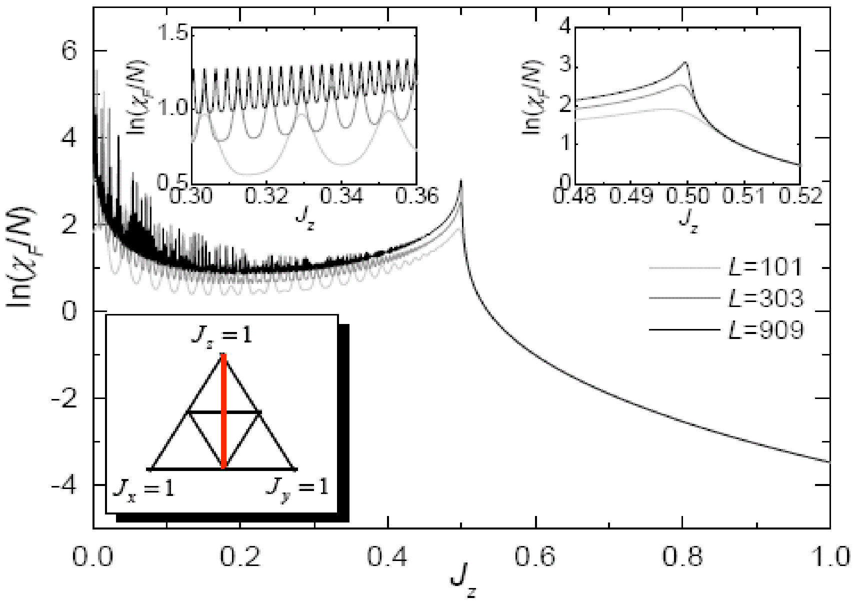
$$
E_0 = -\sum_{\mathbf{q}} \sqrt{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}.
$$

$$
F^2 = \prod_{\mathbf{q}} \frac{1}{2} \left(1 + \frac{\Delta_{\mathbf{q}}\Delta'_{\mathbf{q}} + \epsilon_{\mathbf{q}}\epsilon'_{\mathbf{q}}}{E_{\mathbf{q}}E'_{\mathbf{q}}} \right)
$$

$$
\chi_F \equiv \lim_{\delta\lambda\to 0} \frac{-2\ln F_i}{\delta\lambda^2}
$$

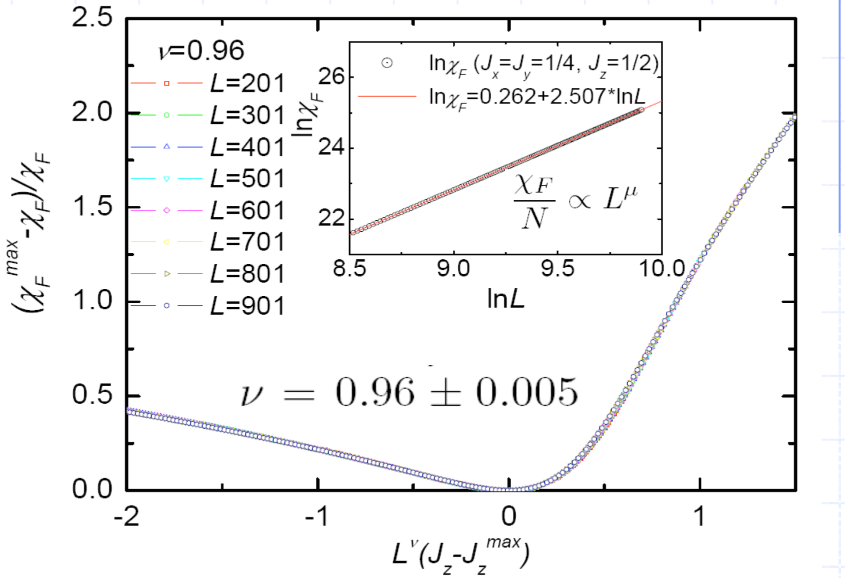
The critical fidelity susceptibility

$$\chi_F = \frac{1}{16}\sum_{\mathbf{q}}\left[\frac{\sin q_x + \sin q_y}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}\right]^2$$



$$\frac{\chi_F}{N} \propto \frac{1}{|J_z - J_z^c|^\alpha}$$

$$\mu = 0.507 \pm 0.0001$$



$$\frac{\chi_F^{\max} - \chi_F}{\chi_F} = f[L^\nu (J_z - J_z^{\max})]$$

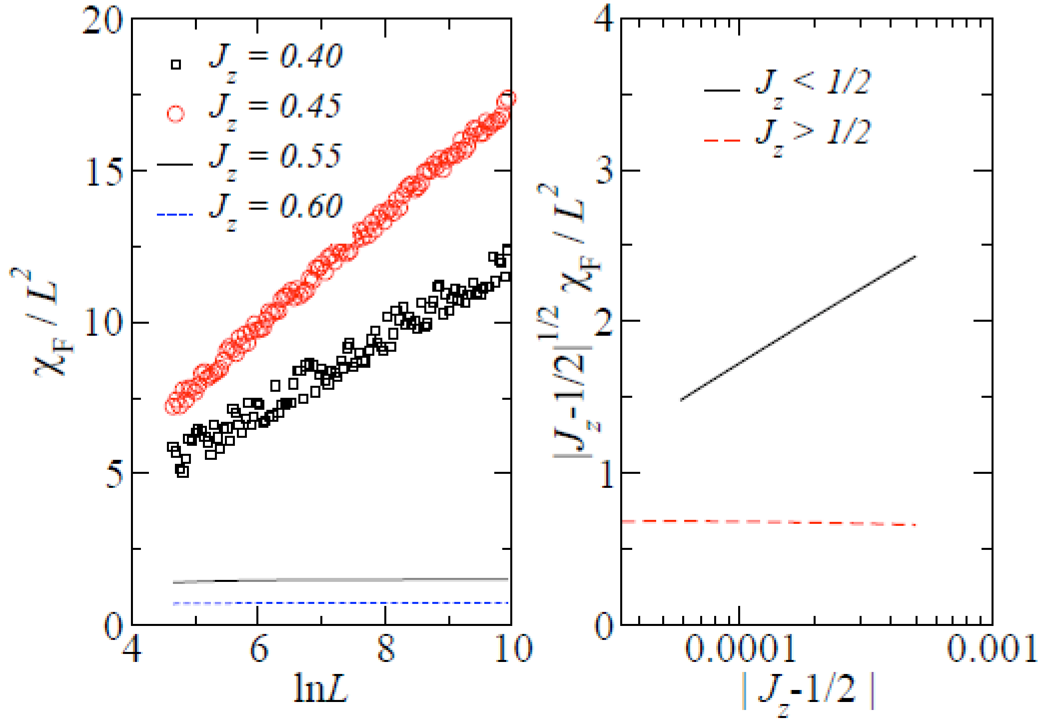
$$\alpha = \frac{\mu}{\nu} = 0.528 \pm 0.001$$

S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A **78**, 012304 (2008).

The critical fidelity susceptibility

$$\chi_F = \frac{1}{16}\sum_{\mathbf{q}}\left[\frac{\sin q_x + \sin q_y}{\epsilon_{\mathbf{q}}^2 + \Delta_{\mathbf{q}}^2}\right]^2$$

Gu and Lin, arXiv: 08073491



$J_z > 1/2$

$$\frac{\chi_F}{L^2} \sim \frac{1}{|J_z - J_{z,c}|^{1/2}}$$

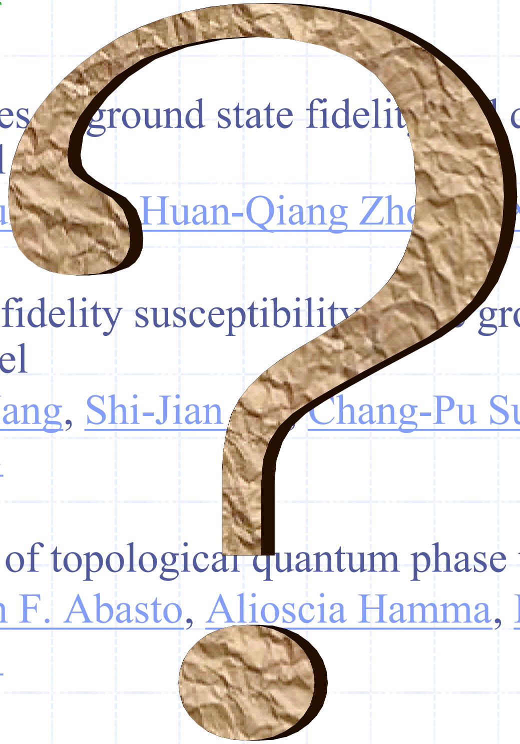
$J_z < 1/2$

$$\frac{\chi_F |J_z - J_{z,c}|^{1/2}}{L^2 \ln L} \sim \ln|J - J_{z,c}|$$

$$\zeta = 1$$

The first conclusion on topological phase transition

- **The fidelity susceptibility can be used to witness the topological quantum phase transition in the Kitaev model**



Title: Singularities ground state fidelit quantum phase transitions for the Kitaev model
Authors: Jian-Hu Huan-Qiang Zho Xiv:0803.0814

Title: Scaling of fidelity susceptibility ground state of Kitaev honeycomb model
Authors: Shuo Yang, Shi-Jian Chang-Pu Sun, Hai-Qing Lin, arXiv:0803.1292

Fidelity analysis of topological quantum phase transitions
Authors: Damian F. Abasto, Alioscia Hamma, Paolo Zanardi, arXiv:0803.2243

The bond-bond correlation

Fidelity susceptibility W. L. You, Y. W. Li, and S. J. Gu, Phys. Rev. E **76**, 022101 (2007).

$$\chi_F = \int \tau \left[\langle \Psi_0 | H_I(\tau) H_I(0) | \Psi_0 \rangle - \langle \Psi_0 | H_I | \Psi_0 \rangle^2 \right] d\tau$$

Bond-bond correlation function

$$\begin{aligned} C(\mathbf{r}_1, \mathbf{r}_2) &= \left\langle \sigma^z_{\mathbf{r}_1,1} \sigma^z_{\mathbf{r}_1,2} \sigma^z_{\mathbf{r}_2,1} \sigma^z_{\mathbf{r}_2,2} \right\rangle \\ &\quad - \left\langle \sigma^z_{\mathbf{r}_1,1} \sigma^z_{\mathbf{r}_1,2} \right\rangle \left\langle \sigma^z_{\mathbf{r}_2,1} \sigma^z_{\mathbf{r}_2,2} \right\rangle \end{aligned}$$

The bond-bond correlation

S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A **78**, 012304 (2008).

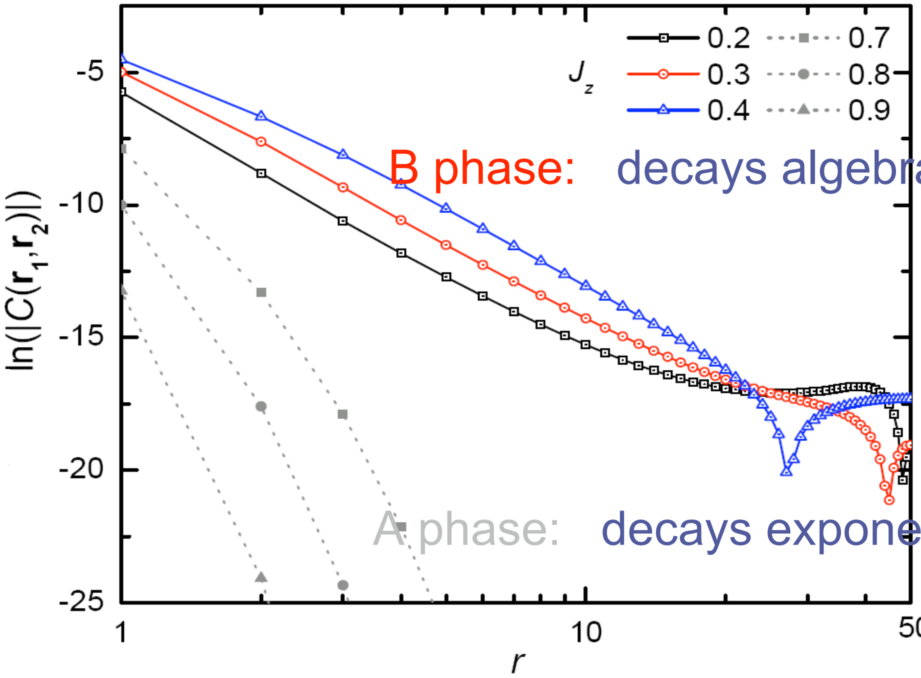
$$\langle \sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\rangle = \langle \sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\rangle = \frac{1}{N}\sum_{\mathbf{q}}\frac{\epsilon_{\mathbf{q}}}{E_{\mathbf{q}}}$$

$$\langle\Psi_0|\,\sigma^z_{\mathbf{r}_1,1}\sigma^z_{\mathbf{r}_1,2}\sigma^z_{\mathbf{r}_2,1}\sigma^z_{\mathbf{r}_2,2}\,|\Psi_0\rangle$$

$$= \frac{1}{N^2}\sum_{\mathbf{q},\mathbf{q}'}\{\cos[(\mathbf{q}-\mathbf{q}')(\mathbf{r}_1-\mathbf{r}_2)]-1\} \times \frac{(\Delta_{\mathbf{q}}\Delta_{\mathbf{q}'}-\epsilon_{\mathbf{q}}\epsilon_{\mathbf{q}'})}{E_{\mathbf{q}}E_{\mathbf{q}'}}$$

$$C(\mathbf{r}_1,\mathbf{r}_2) \propto \frac{1}{|\mathbf{r}_1-\mathbf{r}_2|^4}$$

$$\frac{1}{\xi} = 2\sinh^{-1}\frac{\sqrt{2J_z-1}}{1-J_z}$$



B phase: decays algebraically

A phase: decays exponentially

Quantum criticality in terms of fidelity susceptibility

Model		μ	ν	d^+	α^+	d^-	α^-
1D Ising model($h_c = 1$)	[1]	2	1	1	1	1	1
Lipkin-Meshkov-Glick model($h_c = 1$)	[2]	4/3	2/3	0	2	1	1/2
Kitaev honeycomb model($J_{z,c} = 1/2$)	[3,4]	2.50	1	2	1	2+ln	1/2-ln
Deformed Kitaev toric model [$\lambda_c = \frac{1}{2}\ln(\sqrt{2}+1)$]	[5]	ln	1	1	ln	1	ln
1D AHM ($t_c = 0.456$ for $n = 2/3$)	[6]	5.3	2.65	-	-	1	1.6
Luttinger model($\lambda_c = 1$ of XXZ model)	[7]	-	-	-	-	-	1
Luttinger model($\lambda_c = -1$ of XXZ model)	[7]	-	-	-	1	-	-

1.P. Zanardi and N. Paunkovic, Phys. Rev. E 74, 031123 (2006).
2.H. M. Kwok, W. Q. Ning, S. J. Gu, and H. Q. Lin, Phys. Rev. E 78, 032103 (2008).
3.S. Yang, S. J. Gu, C. P. Sun, and H. Q. Lin, Phys. Rev. A 78, 012304 (2008).
4.S. J. Gu and H. Q. Lin, arXiv:0807.3491.
5.D. F. Abasto, A. Hamma, and P. Zanardi, Phys. Rev. A 78, 010301(R) (2008).
6.S. J. Gu, H. M. Kwok, W. Q. Ning, and H. Q. Lin, Phys. Rev. B 77, 245109 (2008).
7.M. F. Yang, Phys. Rev. B 76, 180403 (R) (2007).

Content

I. Introduction: quantum phase transition, fidelity in quantum information

II. Fidelity and dynamic structure factor in the ground state

III. Fidelity susceptibility and universality class

IV. Fidelity susceptibility in topological phase transitions.

V. Beyond the pure-state fidelity: Density-functional fidelity

VI. Summary

Beyond the pure-state fidelity

1. Reduced fidelity (Zhou, Nicola&Pedro, Wang, Gu)
2. Fidelity per site H. Q. Zhou, J. H. Zhao, and B. Li, arXiv:0704.2940;
3. Operator fidelity X. Wang, Z. Sun, and Z. D. Wang, arXiv:0803.2940
4. Density-functional fidelity (Gu, preprint)

The density-functional theory

$$\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{H}_I + \sum_x \mu_x \hat{n}_x$$

the density-functional fidelity (DFF). According to the Hohenberg-Kohn theorems, the ground-state properties of a quantum many-body system is uniquely determined by the density distribution n_x that minimize the functional for the ground-state energy $E_0[n_x]$. Therefore, the distribution n_x captures the most relevant information about the ground-state. Any change in the structure of the wavefunction can be found by calculating the similarity between two density distributions, i.e. the fidelity,

The density-functional fidelity

$$\rho(x_1, x_1') = \mathrm{tr}|\Psi_0(\lambda)\rangle\langle\Psi_0(\lambda)|$$

$$n = \sum n_x |x\rangle\langle x|$$

$$n_x = \langle\tilde{\Psi}_0(\lambda)|\hat{n}_x|\Psi_0(\lambda)\rangle = \langle\Psi_0(\lambda)|(\partial\hat{H}/\partial\mu_x)|\Psi_0(\lambda)\rangle$$

$$F(\lambda, \lambda\prime) = \mathrm{tr}\sqrt{n(\lambda)n(\lambda')}$$

$$F(\lambda, \lambda+\delta\lambda) = 1 - \frac{(\delta\lambda)^2}{2}\chi_F.$$

$$\chi_F = \sum_x \frac{1}{4n_x}\left(\frac{\partial n_x}{\partial\lambda}\right)^2$$

Example: Hubbard model

$$H = -\sum_{\sigma,j} (c^+_{j,\sigma} c_{j+1,\sigma} + c^+_{j+1,\sigma} c_{j,\sigma}) + U \sum_j n_{j,\uparrow} n_{j,\downarrow}$$

$$2\pi I_j = k_j L - 2\sum_{a=1}^{M} \tan^{-1}\left(\frac{\lambda_a - \sin k_j}{U/4}\right)$$

$$2\pi J_a = 2\sum_{j=1}^{N} \tan^{-1}\left(\frac{\lambda_a - \sin k_j}{U/4}\right) - 2\sum_{b=1}^{M} \tan^{-1}\left(\frac{\lambda_a - \lambda_b}{U/2}\right)$$

$$E = -2\sum_{j=1}^{N} \cos k_j$$

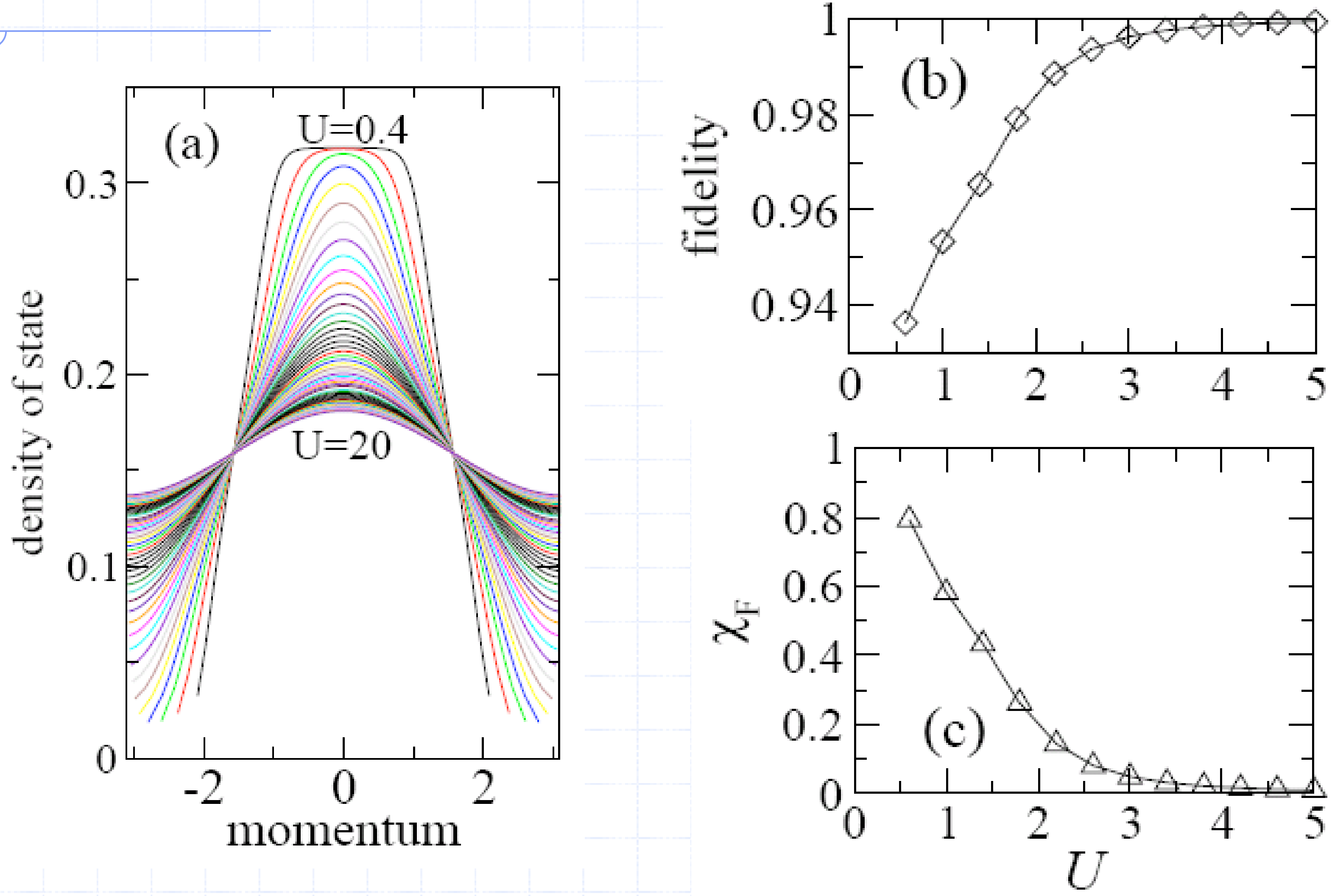
$$I_j = -\frac{N-1}{2}, -\frac{N-3}{2}, \cdots, \frac{N-1}{2}$$

$$J_a = -\frac{M-1}{2}, -\frac{M-3}{2}, \cdots, \frac{M-1}{2}$$

$$\rho\left(\frac{k_j + k_{j+1}}{2}\right) = \frac{1}{L(k_{j+1} - k_j)}$$

$$\frac{N}{L} = \int \rho(k) dk$$

Example: Hubbard model



Summary

1. We establish a general relation between the fidelity and dynamic structure factor of the driving parameter
2. We can learn the universality class of the critical phenomena from the fidelity susceptibility.
3. Fidelity susceptibility and bond-bond long range correlation can also describe the topological phase transitions.
4. We propose a density-functional fidelity to study the quantum phase transitions

Obrigado

謝謝大家

Thank you