Spin dynamics in a one-dimensional Bose-Hubbard model

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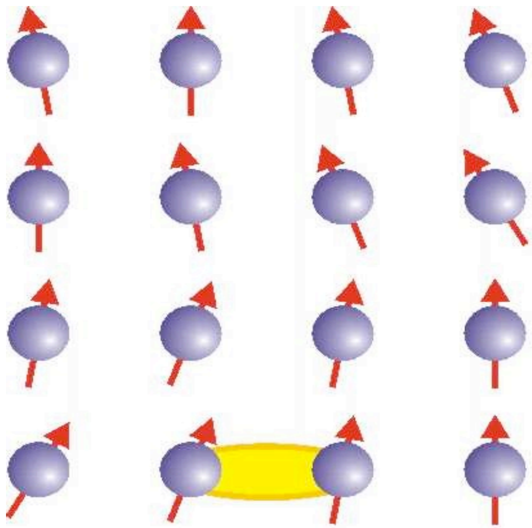
localized vs. itinerant magnetism

Localized $\Longleftrightarrow$ no density fluctuations

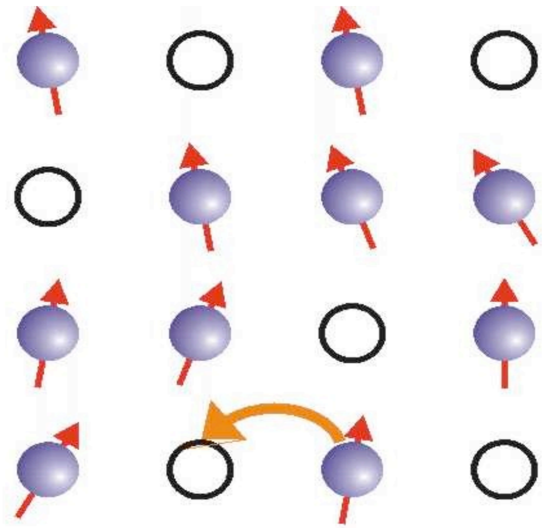


$$H_{ab} = J\,\vec{S}_a \cdot \vec{S}_b$$

$J > 0$ ⇨ magnetization = 0 at zero field: antiferromagnet or paramagnet

$J < 0$ ⇨ magnetization ≠ 0 at zero field: ferromagnet

Itinerant $\Longleftrightarrow$ density fluctuates



$$H = \sum_j \epsilon(\hat{\mathbf{p}}_j) + \sum_{i<j} U(\mathbf{r}_i - \mathbf{r}_j)$$

Q: Is there magnetic order?

Simpler to answer

Q: How do excitations propagate and interact?

More difficult to answer

We are studying dynamical correlation functions of a

1D
itinerant
ferromagnet

Complains:

a) The system does not exist in nature

b) The problem is trivial

c) The problem is already solved

1D itinerant ferromagnetism: an introduction

Spin-1/2 Bose-Hubbard model

Transverse spin-spin correlation function

Spectral function (dynamical structure factor)

Conclusions and perspectives

Itinerant ferromagnetism in 1D: existence in nature

Truly 1D fermionic system cannot be ferromagnetic

[E.Lieb & D. Mattis, Phys. Rev. **125**, 164 (1962)]

! However !

The ground state of a 1D itinerant Bose system is always completely polarized

[see e.g. E. Eisenberg & E. H. Lieb, PRL **89**, 220403 (2002)]

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1D itinerant ferromagnetism appears naturally in 1D ultracold atomic gases

Ultracold atomic gases experiment: 1D, ‘spinless’

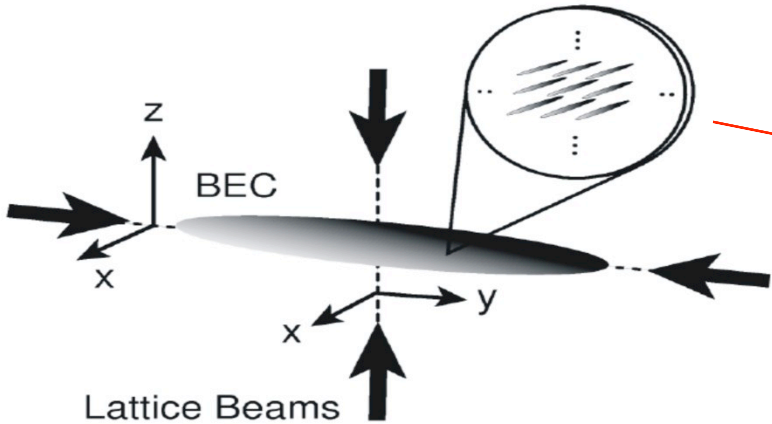
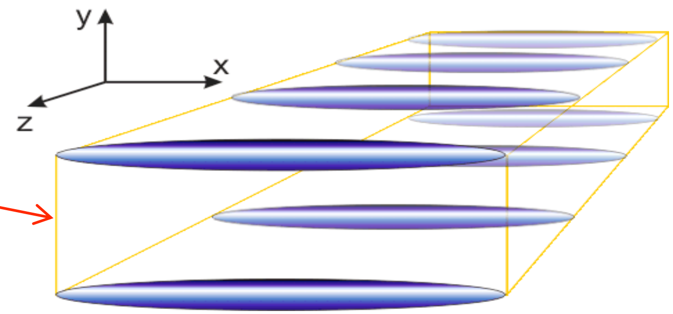


FIG. 1. Schematic setup of the experiment. A 2D lattice potential is formed by overlapping two optical standing waves along the horizontal axis (y axis) and the vertical axis (z axis) with a Bose-Einstein condensate in a magnetic trap. The condensate is then confined to an array of several thousand narrow potential tubes (see inset).



[T. Stoferle *et. al.*, PRL **92**, 130403 (2004)]

[B. Paredes *et. al.*, Nature **429**, 277 (2004)]

[T. Kinoshita *et. al.*, Science **305**, 1125 (2004)]

[T. Kinoshita *et. al.*, PRL **95**, 190406 (2005)]

[B.L. Tolra *et. al.*, PRL **92**, 190401 (2004)]

[… and others …]

Only one transverse band populated ⇨ truly 1D

However! Spin structure is not resolved

Ultracold atomic gases experiment: quasi-1D, spin resolved

[L.E. Sadler *et. al.*, Nature **443**, 164 (2006)]

[J.M. Higbie *et. al.*, PRL **95**, 050401 (2005)]

$^{87}\mathrm{Rb}$ atoms, $F=1$ states

⇩

Spin 1 system

[J.M. McGuirk *et. al.*, PRL **89**, 090402 (2002)]

$|F=1, m_F=-1\rangle$ and $|F=2, m_F=1\rangle$

⇩

Spin $1/2$ system

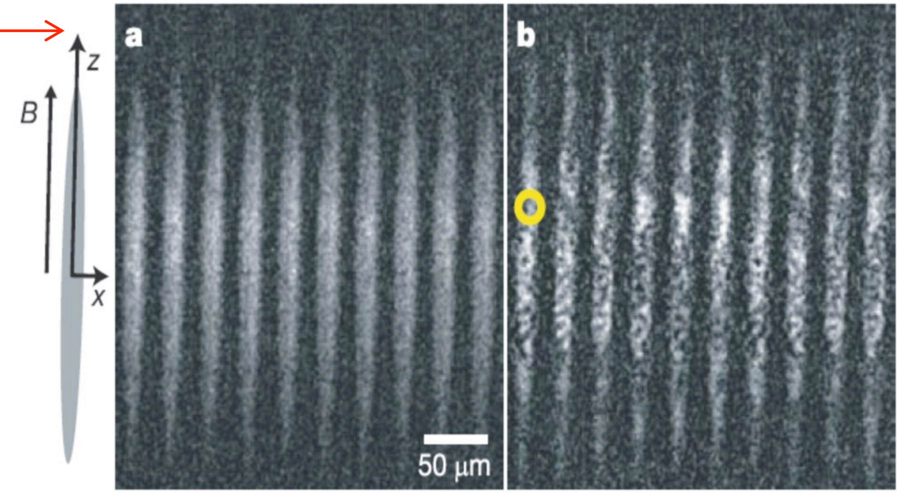


Figure 1 | Direct imaging of inhomogeneous spontaneous magnetization of a spinor BEC. Transverse imaging sequences (first 10 of 24 frames) are shown for a single condensate probed at $T_{\mathrm{hold}} = 36$ ms (**a**) and for a different condensate at $T_{\mathrm{hold}} = 216$ ms (**b**). Shortly after the quench, the

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~ 50 transverse and >1000 longitudinal bands populated ⇨ quasi-1D

Spin structure is resolved

Spin-resolved experiment in truly 1D quantum gases?

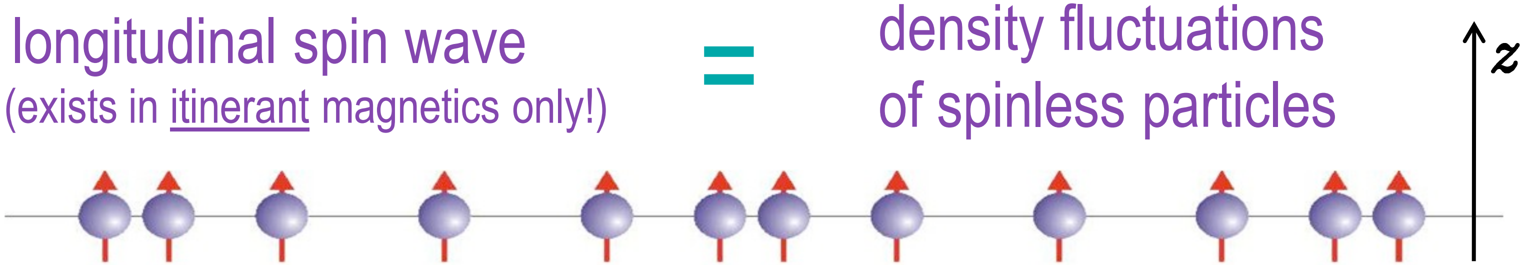
⇨ [A. Widera *et. al.*, PRL **100**, 140401 (2008)]

Why 1D itinerant ferromagnetism is a non-trivial problem?

longitudinal spin wave
(exists in itinerant magnetics only!)

=

density fluctuations
of spinless particles



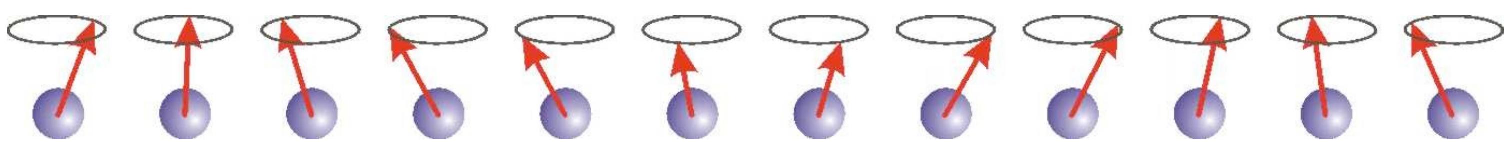
Dynamics is encoded in $G_{\|}(x,t) = \langle \Uparrow | s_z(x,t) s_z(0,0) | \Uparrow \rangle$

$s_j(x), \;\; j = x, y, z$ - local spin operators: $[s_x(x), s_y(x')] = i\delta(x - x') s_z(x)$

transverse spin wave
in itinerant system

$\neq$

transverse spin wave
in localized system



Dynamics is encoded in $G_{\perp}(x,t) = \langle \Uparrow | s_+(x,t) s_-(0,0) | \Uparrow \rangle$

$s_{\pm}(x)$ - spin ladder operators: $s_{\pm} = s_x \pm i s_y$

Longitudinal spin wave dynamics: known how to solve

Longitudinal spin wave = density fluctuations of spinless particles

$$G_{\|}(x,t) = \langle \Uparrow | s_z(x,t) s_z(0,0) | \Uparrow \rangle = \langle \Uparrow | \rho(x,t)\rho(0,0) | \Uparrow \rangle$$

Spinless particles interacting through two-body potential:

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2m} + \sum_{i<j} U(x_i - x_j)$$

---

low energy, momentum:
linear excitation spectrum
⇨ sound waves velocity v

⇨ Bosonization
= Luttinger Liquid theory
= CFT with $c = 1$

⇩

Eq. of motion for LSW= wave equation:

$$\frac{\partial^2 s_z(x,t)}{\partial t^2} - v^2 \frac{\partial^2 s_z(x,t)}{\partial x^2} = 0$$

Longitudinal correlations:
- power-law decay

$$\boxed{G_{\|}(x,t) \sim \frac{x^2 + v^2 t^2}{(x^2 - v^2 t^2)^2}, \qquad x, t \to \infty}$$

Transverse spin dynamics: localized case is trivial

Localized ferromagnetic (Heisenberg model): $H = J \sum_n \vec{s}_n \cdot \vec{s}_{n+1}$ $J < 0$

Ground state $|\Uparrow\rangle$ fully polarized along z Spin wave: $|q\rangle = \sum_n e^{iqn} s_n^- | \Uparrow\rangle$

Spectrum: $H|q\rangle = \epsilon(q)|q\rangle$ ⇨ $\boxed{\epsilon(q) = |J|(1 - \cos q) \sim \frac{q^2}{2m_*}, \qquad q \to 0}$

Effective mass $m_* = \frac{1}{|J|}$

$$G_\perp^H(x,t) = \langle \Uparrow | s_+(x,t) s_-(0,0) | \Uparrow \rangle \qquad x = na$$

$\leftarrow$ Lattice constant

Asymptotically: free particle in the parabolic band $\boxed{G_\perp^H(x,t) \sim \frac{1}{\sqrt{t}} e^{\frac{i m_* x^2}{2t}} \qquad x, t \to \infty}$

Transverse spin dynamics in the itinerant ferromagnet, why non-trivial?

Low-energy dispersion in the itinerant ferromagnetic: like in localized
(supported by the Feynmann single-mode approximation + Bethe Ansatz solutions):

$$\boxed{\varepsilon(k) \simeq \frac{\hbar^2 k^2}{2m}, \qquad k \to 0}$$

However!!!

Exciting transverse spin wave one cannot ignore longitudinal (density) fluctuations

$\Updownarrow$

No factorization of an arbitrary excitation into longitudinal and transverse parts

And so what?

1D itinerant ferromagnetic: not a Luttinger Liquid

Luttinger Liquid for systems with spin: describes antiferro and paramagnetics

Low-energy charge <u>and</u> spin fluctuations are sound waves (spectrum is linear)

⇩

Spin-charge (charge means density) separation: $H \rightarrow H_{eff} = H_{spin} + H_{charge}$

$[H_{spin}, H_{charge}] = 0 \qquad \mathcal{O} = \mathcal{O}_{spin}\mathcal{O}_{charge}$ Free boson Hamiltonians (linear spectrum)

⇨ All Green's functions can be calculated, <u>demonstrate power-law decay</u>

Itinerant ferromagnetic:

Low-energy spin fluctuations are not sound waves (spectrum is quadratic)

⇨ Luttinger liquid description is not applicable

⇨ No spin-charge separation.

On the other hand, due to density fluctuations, it is hardly possible that transverse spin dynamics is the same as that of the localized ferromagnet

???

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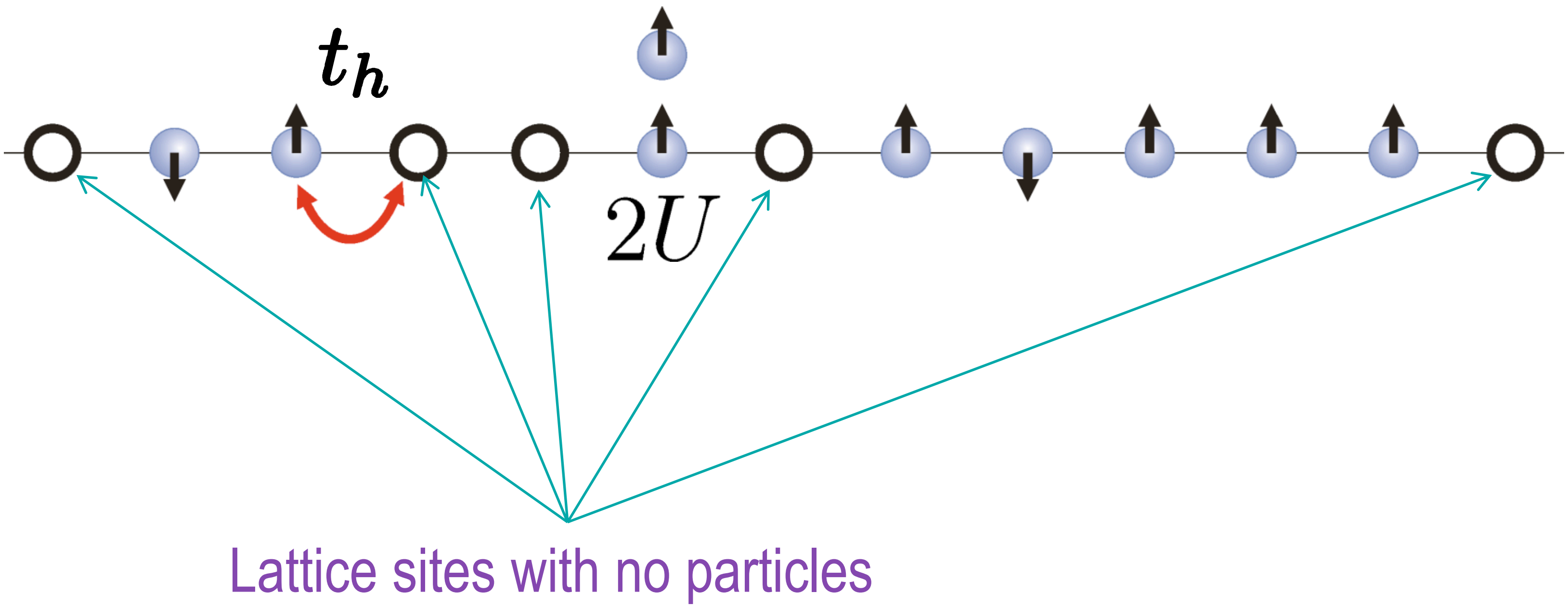
Spin-1/2 Bose-Hubbard model

A system of particles with spin $s = 1/2$ on a 1D lattice with nearest neighbor hoping and on-site repulsion:



t_h - hopping matrix element

$U > 0$ - on-site repulsion

Hamiltonian

The Fock state is generated by Bose fields $b_{\sigma,j}$, $b^\dagger_{\sigma,j}$, where $j = 1, \ldots, M$ and $\sigma = \uparrow, \downarrow$ The Hamiltonian is

$$H = T + V$$

where the kinetic term is

$$T = -t_h \sum_{j=1}^{M} \sum_{\sigma=\uparrow,\downarrow} (b^\dagger_{\sigma j} b_{\sigma j+1} + \mathrm{h.c.})$$

and the potential term is

$$V = U \sum_{j=1}^{M} \varrho_j(\varrho_j - 1) \qquad \varrho_j = \varrho_{\uparrow j} + \varrho_{\downarrow j}$$

2 parameters: $\left\{ \begin{array}{l} \text{Hopping/on-site interaction} \\ \text{Filling factor} \end{array} \right.$

Particular case: filling factor close to zero

Filling factor $\nu = \frac{N}{M}$ ← Number of particles, ← Number of sites

Lattice spacing

Low filling limit: $\nu \to 0$ $\quad a \to 0$ $\quad M \to \infty$ with $\begin{cases} \nu/a = \rho_0 \\ Ma = L \end{cases}$

Spin-1/2 Bose-Hubbard model ⇨ Gaudin-Yang model

(= Lieb-Liniger model with spin)

$$H = \int_0^L dx \left[-\frac{1}{2m} \sum_{\sigma=\uparrow,\downarrow} \psi_\sigma^\dagger(x) \partial_x^2 \psi_\sigma(x) + g : \varrho(x)^2 : \right]$$

where $\frac{1}{2m} = a^2 t_h$ $\quad g = aU$ $\quad$ and $\quad [\psi_{\sigma x}, \psi^\dagger_{\sigma' x'}] = \delta_{\sigma\sigma'} \delta(x - x')$

⇩

Spin dynamics

$\langle \Uparrow | s_+(x,t) s_-(0,0) | \Uparrow \rangle$

M. B. Zvonarev, V. V. Cheianov, and T. Giamarchi, PRL **99**, 240404 (2007)

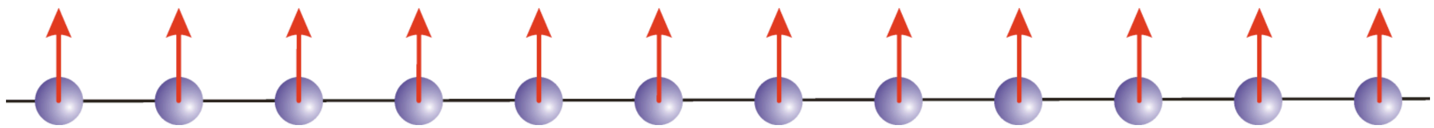
S. Akhanjee and Y. Tserkovnyak, Phys. Rev. B **76,** 140408 (2007)

K. A. Matveev and A. Furusaki, PRL **101**, 170403 (2008)

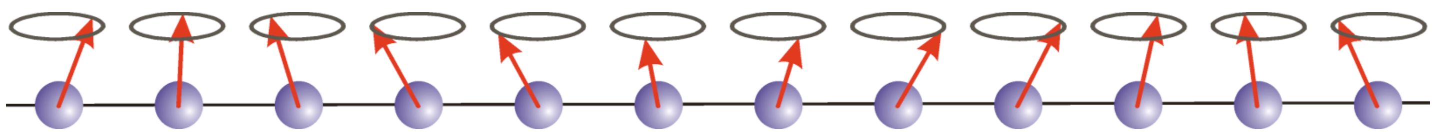
A. Kamenev and L.I. Glazman, arXiv:0808.0479

Particular case: filling factor equal to one

For filling factor $\nu = 1$ and large enough U/t_h the system is incompressible.
(numerics predicts $U/t_h > 4.3$) ⇨ The ground state is:



Low-energy excitations are transverse spin waves:



For the Heisenberg Hamiltonian $H = -2J\sum_j \vec{s}(j)\cdot\vec{s}(j+1)$

transverse spin-spin Green's function $G_\perp(j,t) = \langle \Uparrow | s_+(j,t) s_-(0,0) | \Uparrow \rangle$

is given by $G_\perp(j,t) = e^{-i\frac{\pi}{2}j} e^{2iJt} \mathcal{J}_j(2Jt)$

where $\mathcal{J}_j(2Jt)$ is the Bessel function of the first kind

t-J approximation

$$H = T + V \qquad T = -t_h \sum_{j=1}^{M} \sum_{\sigma=\uparrow,\downarrow} (b^\dagger_{\sigma j} b_{\sigma j+1} + \mathrm{h.c.}) \qquad V = U \sum_{j=1}^{M} \varrho_j (\varrho_j - 1)$$

⇨ t-J approximation: $V \gg T$ ⇨ multiple occupancy is excluded

Denote by $\mathcal{P}$ the projector onto the space of excluded multiple occupancy. Then the second order perturbation theory gives

$$H_{tJ} = \mathcal{P} T \mathcal{P} - \sum_a \mathcal{P} \frac{T|a\rangle\langle a|T}{E_a} \mathcal{P}$$

Are there collective variables in which spin and charge dynamics separate?

Nested variables

H_{tJ} can be written through spinless fermions $c_j, c_j^\dagger$ + nested spin $\vec{\ell}(\mathcal{N}_j)$

$$H_{tJ} = T + \frac{t_h^2}{2U}\sum_j Q_j[\vec{\ell}(\mathcal{N}_j) + \vec{\ell}(\mathcal{N}_j+1)]^2$$

where

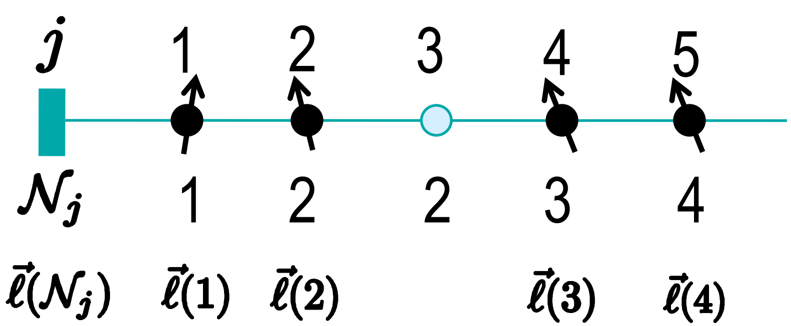
$$Q_j = c_j^\dagger c_{j-1}^\dagger c_{j+1} c_j + c_{j+2}^\dagger c_{j+1}^\dagger c_{j+1} c_j + 2c_{j+1}^\dagger c_j^\dagger c_{j+1} c_j$$

and

$$\vec{\ell}(\mathcal{N}_j) = \sum_m \vec{\ell}(m) \underbrace{\int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-i\lambda(m-\mathcal{N}_j)}}_{\delta_{m,\mathcal{N}_j}} \qquad \mathcal{N}_j = \sum_{n\le j} c_n^\dagger c_n$$

$\mathcal{N}_j$ labels particles on a lattice:

Labeling particles: hard wall boundary conditions for a finite system + labeling from left to right



Nested variables (continued)

Filling factor slightly below one, $1-\nu \ll 1$ ⇨ neglect charge fluctuations:

$$Q_j = \cancel{c_j^\dagger c_{j-1}^\dagger c_{j+1} c_j} + \cancel{c_{j+2}^\dagger c_{j+1}^\dagger c_{j+1} c_j} + 2c_{j+1}^\dagger c_j^\dagger c_{j+1} c_j \longrightarrow -2$$

$$H_{tJ} = T + \frac{t_h^2}{2U} \sum_j Q_j [\vec{\ell}(\mathcal{N}_j) + \vec{\ell}(\mathcal{N}_j + 1)]^2 \qquad \vec{\ell}(\mathcal{N}_j) \longrightarrow \vec{\ell}(j)$$

⇨ Spin and charge dynamics in H_{tJ} separate:

$$H_{tJ} = \underbrace{-t_h \sum_j (c_j^\dagger c_{j+1} + \text{h.c.})}_{\text{Free fermions}} \underbrace{- 2J \sum_m \vec{\ell}(m) \cdot \vec{\ell}(m+1)}_{\text{Heisenberg ferromagnet}} \qquad J = \frac{t_h^2}{U} \xleftarrow{} \nu = 1$$

Arbitrary filling: spin and charge dynamics still separate in H_{tJ}

$$J = \frac{t_h^2}{2\pi U}(2\pi\nu - \sin 2\pi\nu)$$

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Spin-spin transverse Green's function in nested variables

$$H_{tJ} = -t_h \sum_j (c_j^\dagger c_{j+1} + \mathrm{h.c.}) - 2J \sum_m \vec{\ell}(m) \cdot \vec{\ell}(m+1)$$

We factorize the ground state: $|\Uparrow\rangle = |FS\rangle \otimes |\Uparrow\rangle_H$

We factorize local spin operators: $\vec{s}(j) = \varrho_j \vec{\ell}(\mathcal{N}_j) \simeq \vec{\ell}(\mathcal{N}_j)$

$$\Rightarrow \quad \boxed{G_\perp(j,t) = \langle \Uparrow | s_+(j,t) s_-(0,0) | \Uparrow \rangle = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} G_H(\lambda,t) D_\nu(\lambda; j,t)}$$

where G_H is a Green's function of the Heisenberg ferromagnet,

$$G_H(\lambda,t) = \sum_j e^{-i\lambda j}\,{}_H\langle \Uparrow | \ell_+(n,t) \ell_-(0,0) | \Uparrow \rangle_H = e^{-2iJt(1-\cos\lambda)}$$

and D_ν is a Green's function of free spinless fermions,

$$D_\nu(\lambda; j,t) = \langle \mathrm{FS} | e^{i\lambda \mathcal{N}_j(t)} e^{-i\lambda \mathcal{N}_0(0)} | \mathrm{FS} \rangle \qquad \mathcal{N}_j = \sum_{n \le j} c_n^\dagger c_n$$

Analysis of $D_\nu(\lambda; j, t)$: Fredholm determinant

$$D_\nu(\lambda; j, t) = \langle \mathrm{FS}| e^{i\lambda \mathcal{N}_j(t)} e^{-i\lambda \mathcal{N}_0(0)} |\mathrm{FS}\rangle$$

Free fermion problem ⇨ use Wick theorem ⇨ represent D_ν as determinant of some infinite-dimensional matrix (Fredholm determinant)

>>> <<<

Can be investigated analytically

V. E. Korepin, N. M. Bogoliubov, and A. G. Izergin, QISM and Correlation Functions, Cambridge University Press, (1993)

F. Göhmann, A.G. Izergin, V.E. Korepin, A.G. Pronko, Int. J. Mod. Phys. B **12**, 2409 (1998)

V.Cheianov and M. Zvonarev, J. Phys. A: Math. Gen. **37**, 2261 (2004)

and numerically

Analysis of $D_\nu(\lambda; j, t)$: bosonization

Filling factor slightly below one, $1 - \nu \ll 1$ ⇨ characteristic scales (large):

$$q_F = \pi(1-\nu) \qquad E_F = t_h q_F^2 \qquad t_F = E_F^{-1}$$

If $j > q_F^{-1}$ or $t > t_F$ we can use bosonization (Luttinger Liquid)

$$\mathcal{N}_j = \sum_{n \le j} c_n^\dagger c_n \longrightarrow \mathcal{N}_j(t) = \nu j - \frac{1}{\pi}\partial_x \phi(x,t) \qquad x = aj$$

This implies for $D_\nu(\lambda; j, t) = \langle \mathrm{FS}| e^{i\lambda \mathcal{N}_j(t)} e^{-i\lambda \mathcal{N}_0(0)} |\mathrm{FS}\rangle$ the asymptotic expression

$$\boxed{D_\nu(\lambda; j, t) \simeq \exp\left\{ -i\lambda\nu j - \frac{\lambda^2}{4\pi^2} \ln \frac{|j^2 - v_F^2 t^2|}{v_F^2 t_c^2} \right\} \qquad j > q_F^{-1} \text{ or } t > t_F \qquad v_F = 2 q_F t_h \qquad t_c \simeq 5.2 \times 10^{-2} t_F}$$

Analysis of $G_\perp(j,t)$ for $J > E_F$

$$G_\perp(j,t) = \langle \Uparrow | s_+(j,t) s_-(0,0) | \Uparrow \rangle = \int_{-\pi}^{\pi} \frac{d\lambda}{2\pi} G_H(\lambda,t) D_\nu(\lambda;j,t)$$

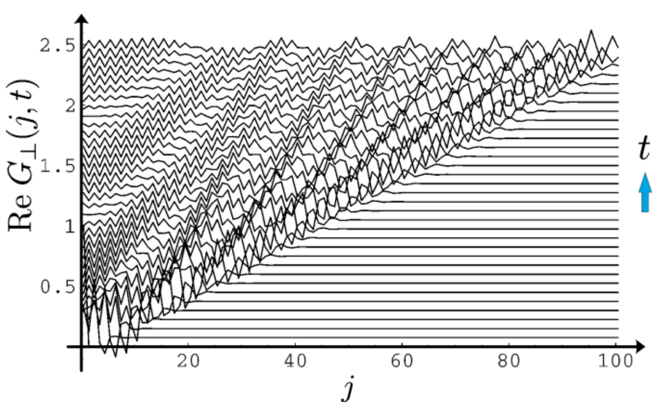
$$G_H(\lambda,t) = e^{-2iJt(1-\cos\lambda)} \qquad D_\nu(\lambda;j,t) \simeq \exp\left\{ -i\lambda\nu j - \frac{\lambda^2}{4\pi^2} \ln \frac{|j^2 - v_F^2 t^2|}{v_F^2 t_c^2} \right\}$$

Two small parameters: $J/t_h \ll 1$ and $E_F/t_h \ll 1$

If $J > E_F$ ⇨ $G_\perp$ behaves like in the undoped system:

$$G_\perp(j,t) = e^{-i\frac{\pi}{2}j} e^{2iJt} \mathcal{J}_j(2Jt)$$

- oscillates for $j < 2Jt$
- decays as $e^{-j \ln j}$ for $j > 2Jt$



dispersion relation $\omega(k) = 2J(1-\cos k)$

⇨ maximal group velocity $v_{\max} = 2J$

Analysis of $G_\perp(j,t)$ for $E_F > J$

If $E_F > J$ then for $t_F < t < J^{-1}$

$$G_\perp(j,t) \simeq \sqrt{\frac{\pi}{2\ln(t/t_c)}}\exp\left\{-\frac{(\pi j)^2}{2\ln(t/t_c)}\right\}$$

logarithmic diffusion: $\langle X^2\rangle \sim \ln t$

! $G_\perp$ does not depend on J ⇨ spin excitations propagate due to fluctuations of holes, and not due to exchange

For $t > J^{-1}$ undoped regime recovers

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The spectral function: threshold behavior

$$G_\perp(j,t) = \int \frac{dk}{2\pi} e^{ikj} \int \frac{d\omega}{2\pi} e^{-i\omega t} A(k,\omega) \longleftarrow$$

spectral function
(dynamical structure factor)

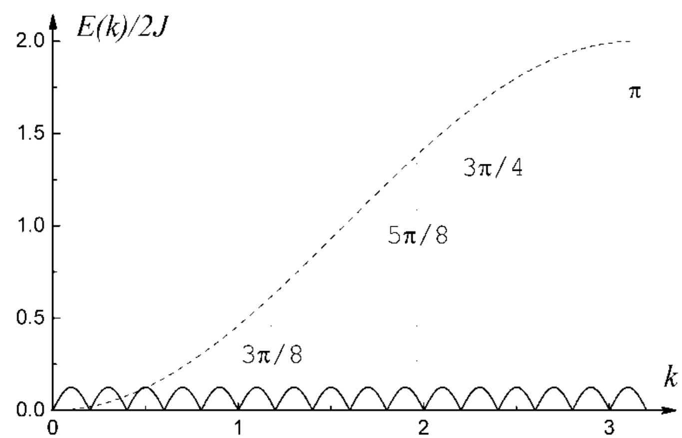
$$A(k,\omega) = \sum_f \delta(\hbar\omega - E_f(k)) |\langle f,k|s_-(k)|\Uparrow\rangle|^2$$

Here $H|f,k\rangle = E_f(k)|f,k\rangle$ and f enumerates all the states with momentum k

Threshold behavior: $A(k,\omega) \sim \theta(\omega - E_s(k/\nu))[\omega - E_s(k/\nu)]^{\Delta(k)}$

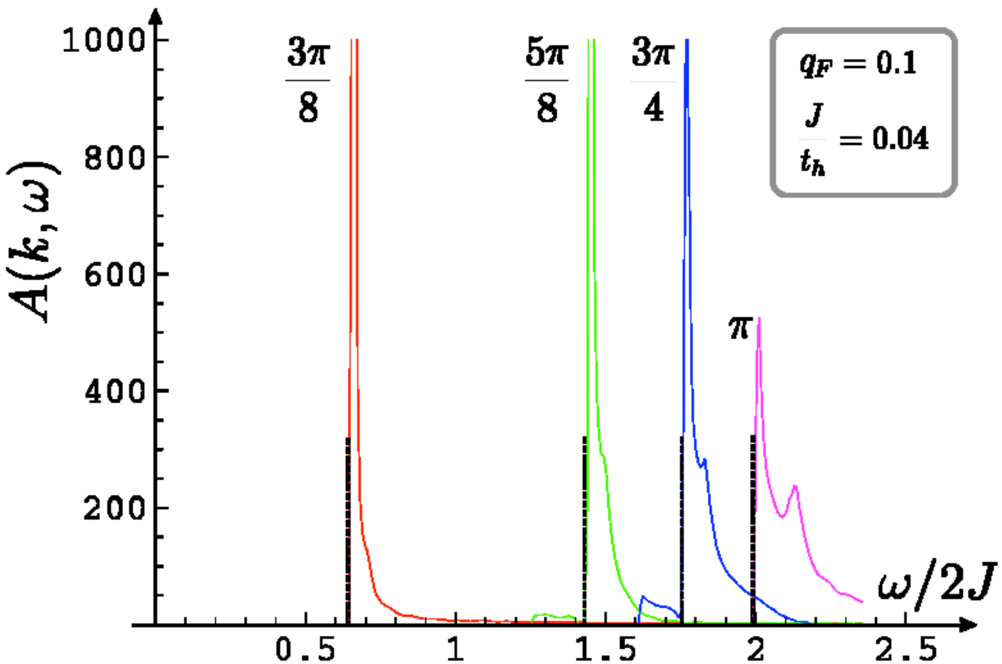
$$\Delta(k) = -1 + \frac{k^2}{2\pi^2}$$

$$E_s(k) = 2Jt(1-\cos k)$$



The spectral function: numerics

By calculating the Fredholm determinant numerically we get:



Conclusions

We studied dynamical properties of 1D spin-1/2 Bose-Hubbard model in the regime of low doping and strong on-site repumsion

We got both exact and asymptotic expressions for the transverse spin-spin Green’s function

The low-energy behavior of the Green’s function is not of Luttinger Liquid type

We hope that the characteristic signatures of the discussed regime could be seen in the experiments with ultracold atomic gases

Perspectives

Arbitrary filling factor?

Arbitrary strength of the on-site repulsion?

Relation to the the problem of the mobile impurity in the Luttinger Liquid

Relation to the the problem of the particle dynamics
in the dissipative environment?