New Directions in Valence Bond Solid Antiferromagnets

or

Quantum Magnetism with Tinker Toys

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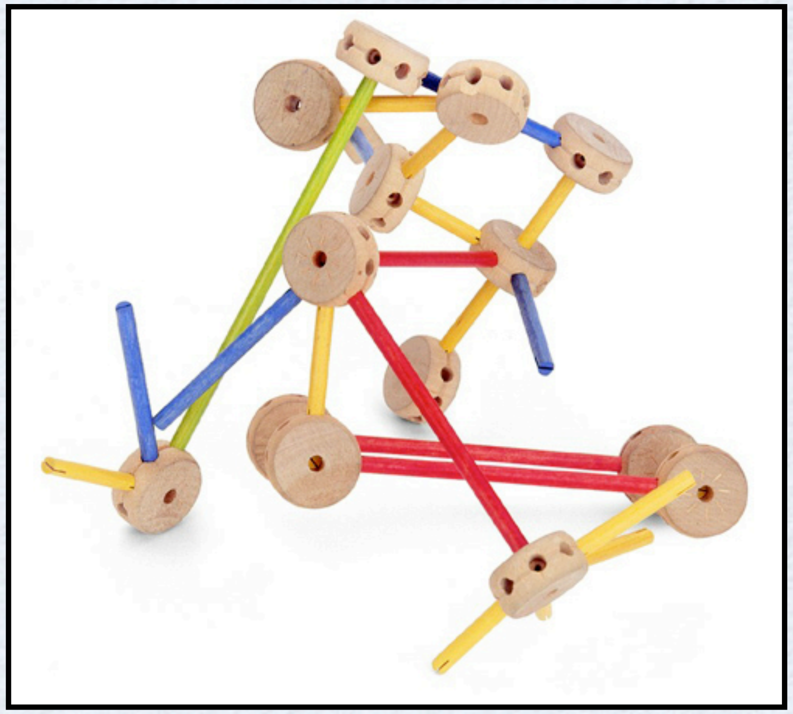


Local interactions prefer local singlets

Heisenberg antiferromagnet : $\mathcal{H} = \boldsymbol{S}_i \cdot \boldsymbol{S}_j$

$$E = \tfrac{1}{2} J(J+1) - S(S+1)$$

where $J \in \{0, 1, \ldots, 2S\}$. Singlet (J=0) has lowest energy.

Lattice Heisenberg antiferromagnet : $\mathcal{H} = \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$

Ground state (Bethe Ansatz) energy per bond (d=1 chain):

$$E_0 = \tfrac{1}{4} - \ln 2 = -0.443147\ldots$$

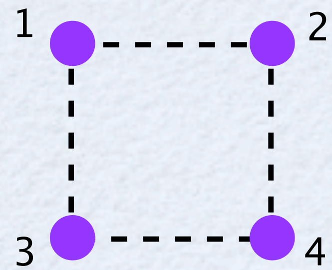
$$E_{\text{singlet}} = -\tfrac{3}{4} = -0.75$$

Not every bond can be a singlet!

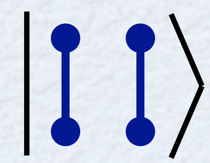
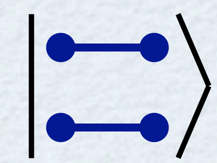
Small antiferromagnetic clusters

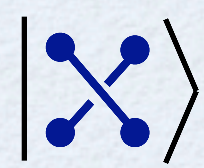
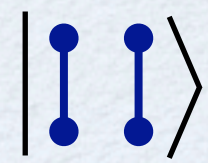
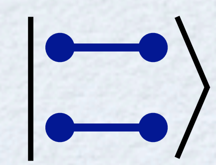
Two spins S = ½: singlet ground state $|\Psi\rangle = 2^{-1/2}\{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}$

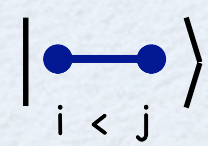
Four spins S = ½:

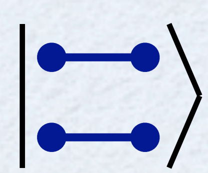
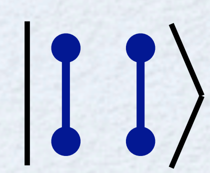


$\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 0 \oplus 1 \oplus 1 \oplus 1 \oplus 2$

Solution: two independent singlets,  and 

(linear dependence:  =  − )

phase convention: $\sqrt{2}$  $= |\uparrow_i \downarrow_j\rangle - |\downarrow_i \uparrow_j\rangle$

ground state: $\sqrt{3}\,|\Psi\rangle =$  + 

"resonating valence bonds" (RVB)

The quantum Néel state

Classical Néel state:

$m = S$

$$|\mathrm{CN}\rangle = \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle$$

selected by $S_i^z S_j^z$

(A/B sublattice)

Quantum Néel state:

$m = S - 1$

$$|\mathrm{QN}\rangle = \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle - \frac{1}{4(z-1)S} \left| \begin{array}{cccc} \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \\ \downarrow & \uparrow & \downarrow & \uparrow \\ \uparrow & \downarrow & \uparrow & \downarrow \end{array} \right\rangle + \ldots$$

reduced due to $S_i^+ S_j^-$

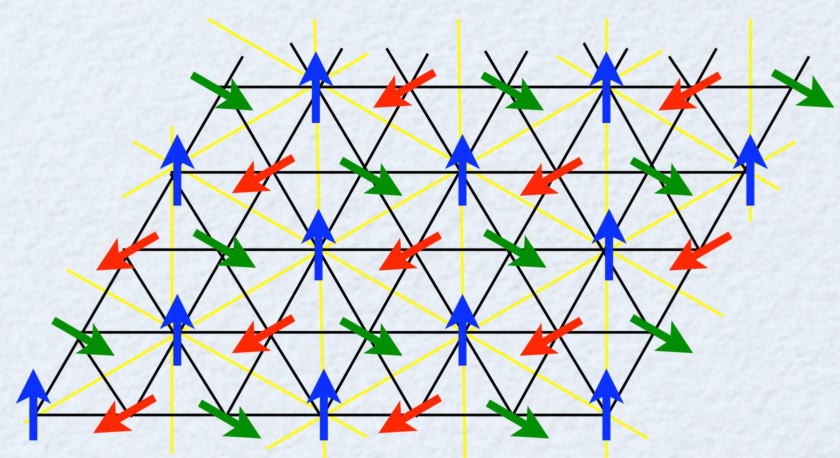
Classical order is reduced due to quantum fluctuations.

E.g. square lattice: $M \simeq 0.31$ for $S = \frac{1}{2}$ (Reger and Young, 1988)

Triangular lattice: three-sublattice

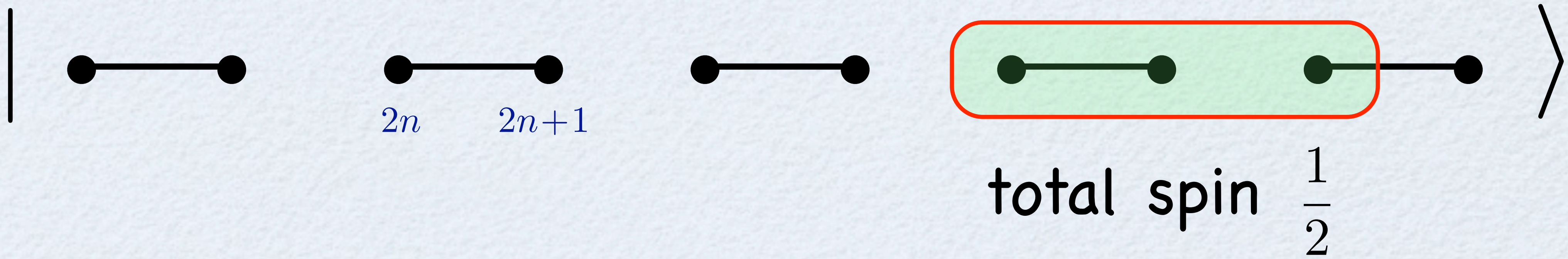
$\sqrt{3} \times \sqrt{3}$ Néel state has $M \simeq 0.20$

(Bernu et al., 1994)



Majumdar-Ghosh model (1969)

(Majumdar and Ghosh, 1969)

$|\Psi_{\mathrm{A}}\rangle =$ 

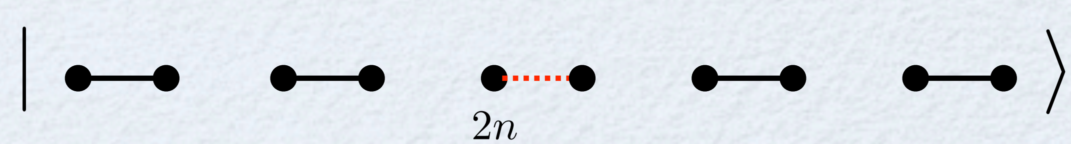
Generally we have $\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2}$ missing quadruplet!

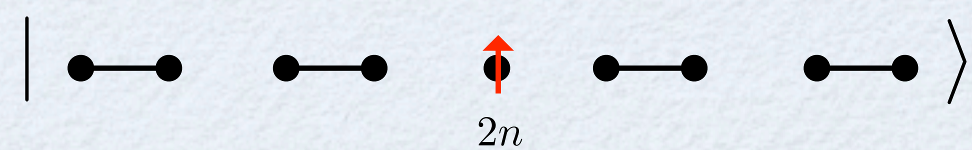
Form projection operator : $(\boldsymbol{S}_i + \boldsymbol{S}_j + \boldsymbol{S}_k)^2 = \begin{cases} 3/4 \\ 15/4 \end{cases}$

$$\Longrightarrow \qquad \mathrm{P}_{3/2}(i,j,k) = \frac{1}{3}\Big[(\boldsymbol{S}_i + \boldsymbol{S}_j + \boldsymbol{S}_k)^2 - \tfrac{3}{4}\Big]$$

$$\mathcal{H} = \sum_n \mathrm{P}_{3/2}(n-1\,,\,n\,,\,n+1) = \tfrac{4}{3}\sum_n \left[\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + \tfrac{1}{2}\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+2}\right] + E_0$$

elementary excitations

$|k\,,\,\mathrm{magnon}\rangle = \sum_n e^{2nik}$ 

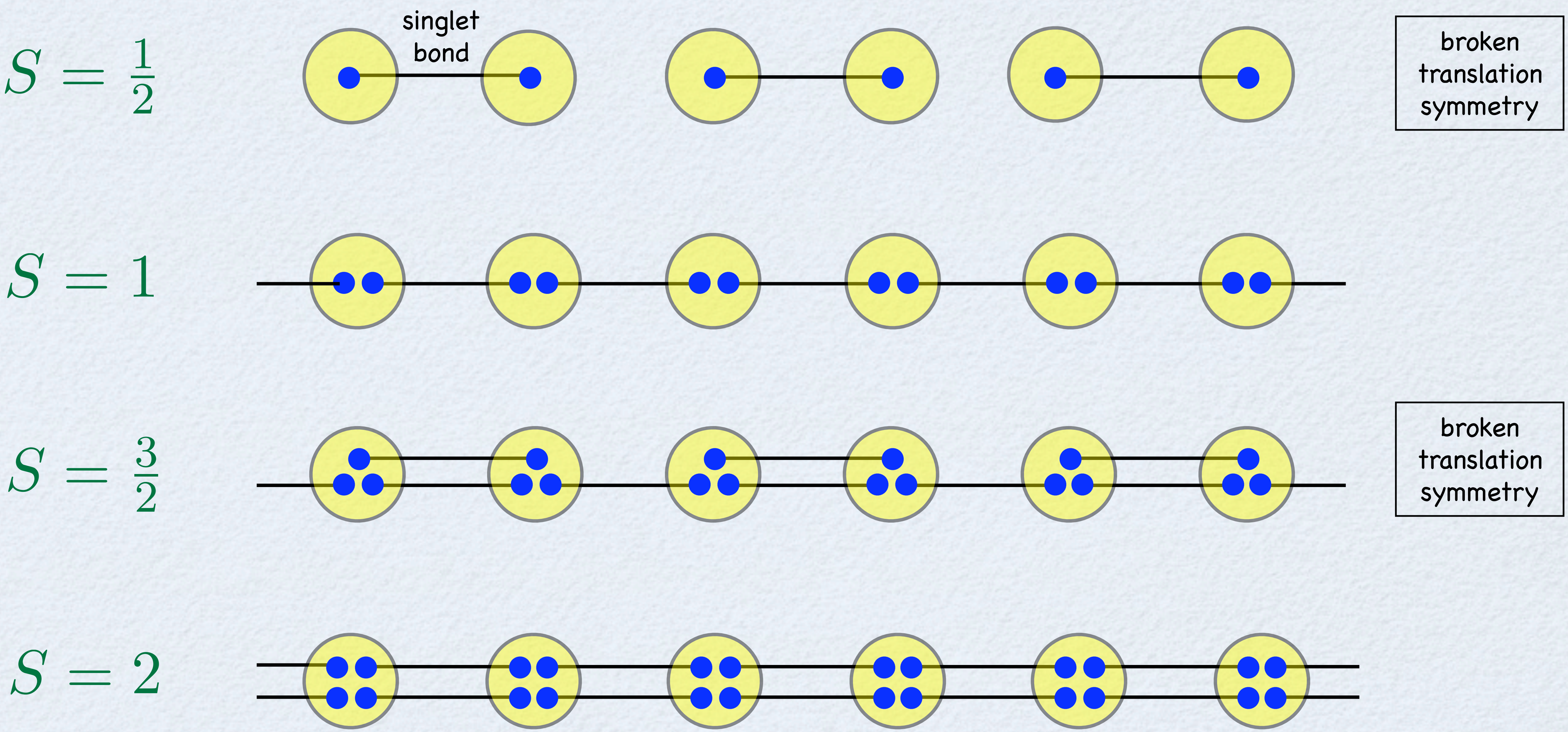
$|k\,,\,\mathrm{soliton}\rangle = \sum_n e^{2nik}$ 

Valence bond solid states

(Affleck, Kennedy, Lieb, and Tasaki, 1987)

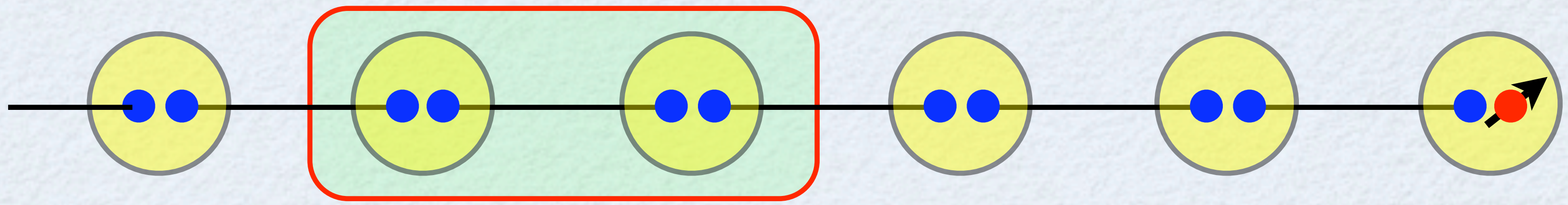
Spin S from symmetrized product of 2S spin-½ :

$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \quad , \quad \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} = \frac{1}{2} \oplus \frac{1}{2} \oplus \frac{3}{2} \quad , \text{ etc.}$$



Projection operators

S=1 AKLT chain :



projector onto total bond spin $|\mathbf{S}_i + \mathbf{S}_j| = J$:

$$\mathrm{P}_J^{[S]}(ij) = \prod_{\substack{k=0 \\ (k \neq J)}}^{2S} \frac{(\boldsymbol{S}_i + \boldsymbol{S}_j)^2 - k(k+1)}{J(J+1) - k(k+1)}$$

this is a polynomial of order 2S in $(\mathbf{S}_i \cdot \mathbf{S}_j)$

projector onto J=2 : $\mathrm{P}_{J=2}^{[S=1]} = \frac{1}{6} + \frac{1}{2}\,\boldsymbol{S}_i \cdot \boldsymbol{S}_j + \frac{1}{3}\big(\boldsymbol{S}_i \cdot \boldsymbol{S}_j\big)^2$

AKLT Hamiltonian : $\mathcal{H} = \sum_n \Big[\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + \frac{1}{3}\big(\boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1}\big)^2\Big]$

bilinear biquadratic

Antiferromagnetic S=1 Heisenberg chain

exchange interaction   anisotropy: prefers $S_n^z = 0$

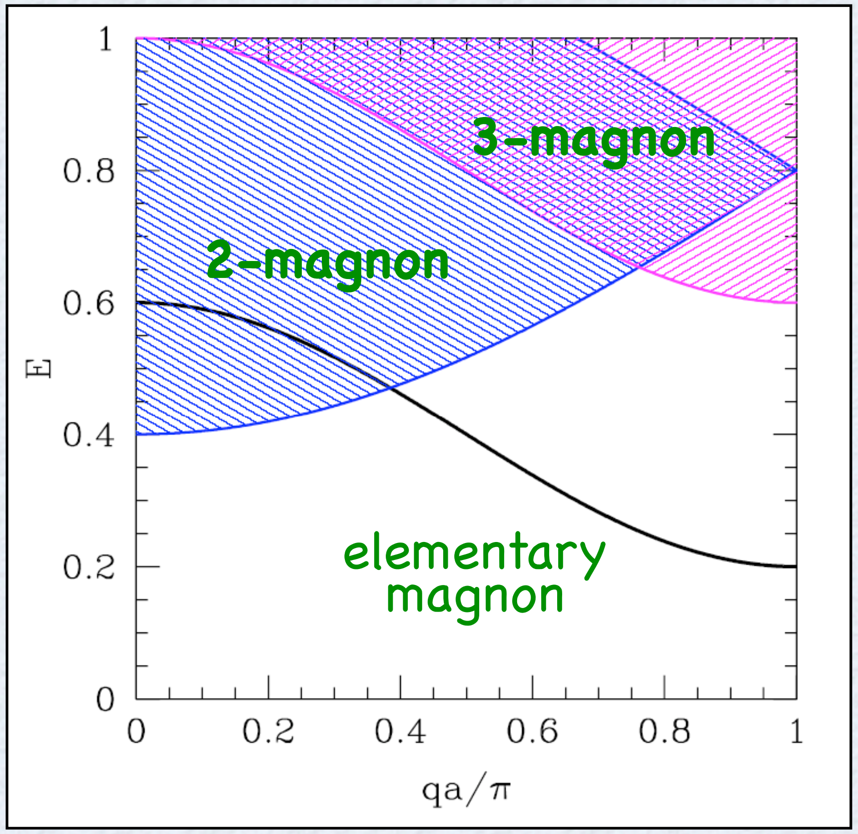
$$\mathcal{H} = J \sum_n \boldsymbol{S}_n \cdot \boldsymbol{S}_{n+1} + D \sum_n (S_n^z)^2$$

The elementary excitation is a triplet (S=1) with dispersion ω(q), with an excitation gap at **q=π**. matrix elements. This is the **Haldane gap**.

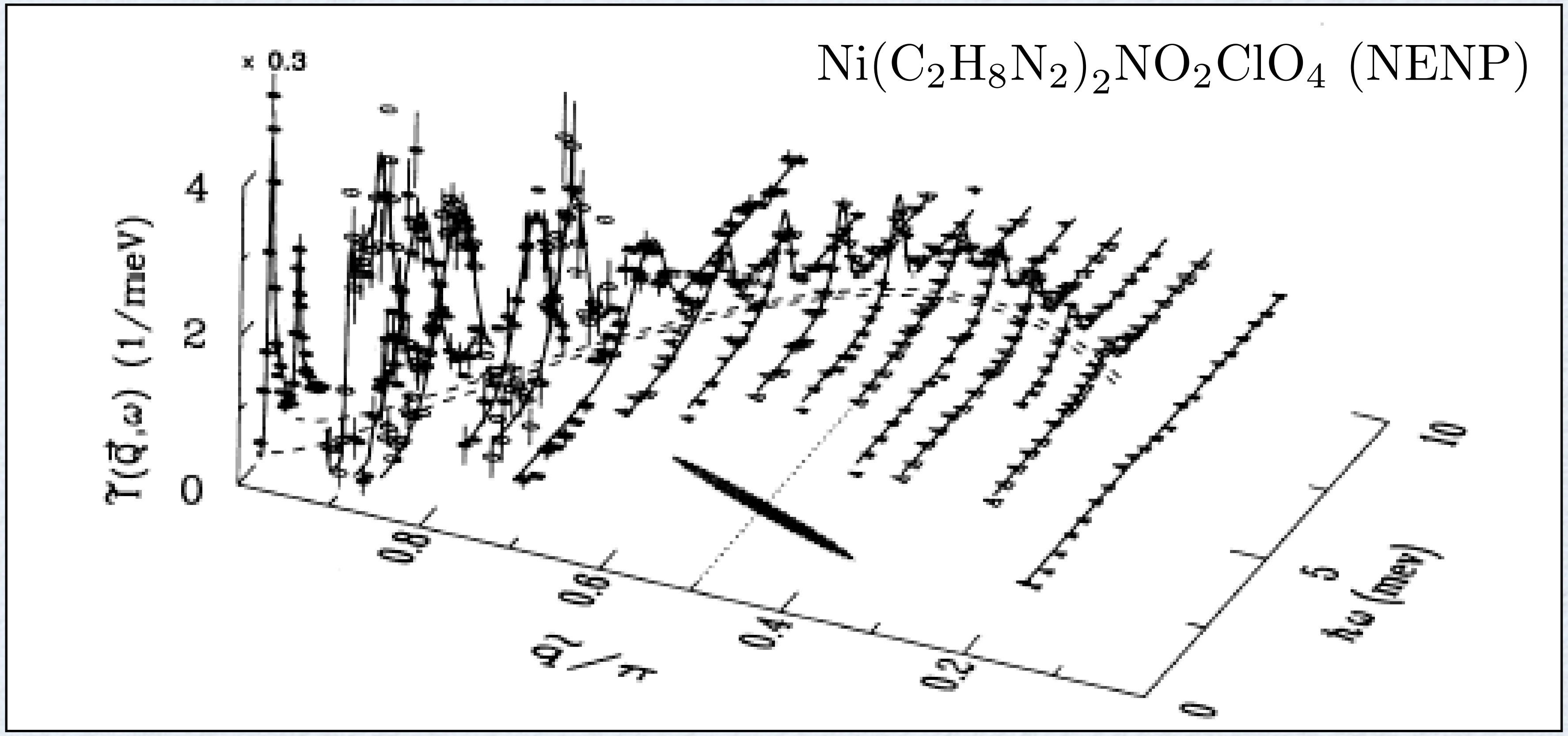
$$\langle \boldsymbol{S}_l \cdot \boldsymbol{S}_{l+n} \rangle \sim (-1)^n \, |n|^{-1/2} \exp\big(-|n|a/\xi\big)$$

F. D. M. Haldane, Phys. Lett. **93A**, 464 (1983)

magnon / multimagnon continua for AKLT chain



S. Ma et al., PRL **69**, 3571 (1992)



Schwinger bosons

DPA and Auerbach, 1988
Read and Sachdev, 1990

Schwinger representation of SU(2):

$$S^+ = a^\dagger b \qquad S^z = \frac{1}{2}(n_a - n_b)$$

$$S^- = a\, b^\dagger \qquad 2S = n_a + n_b$$

Heisenberg interaction:

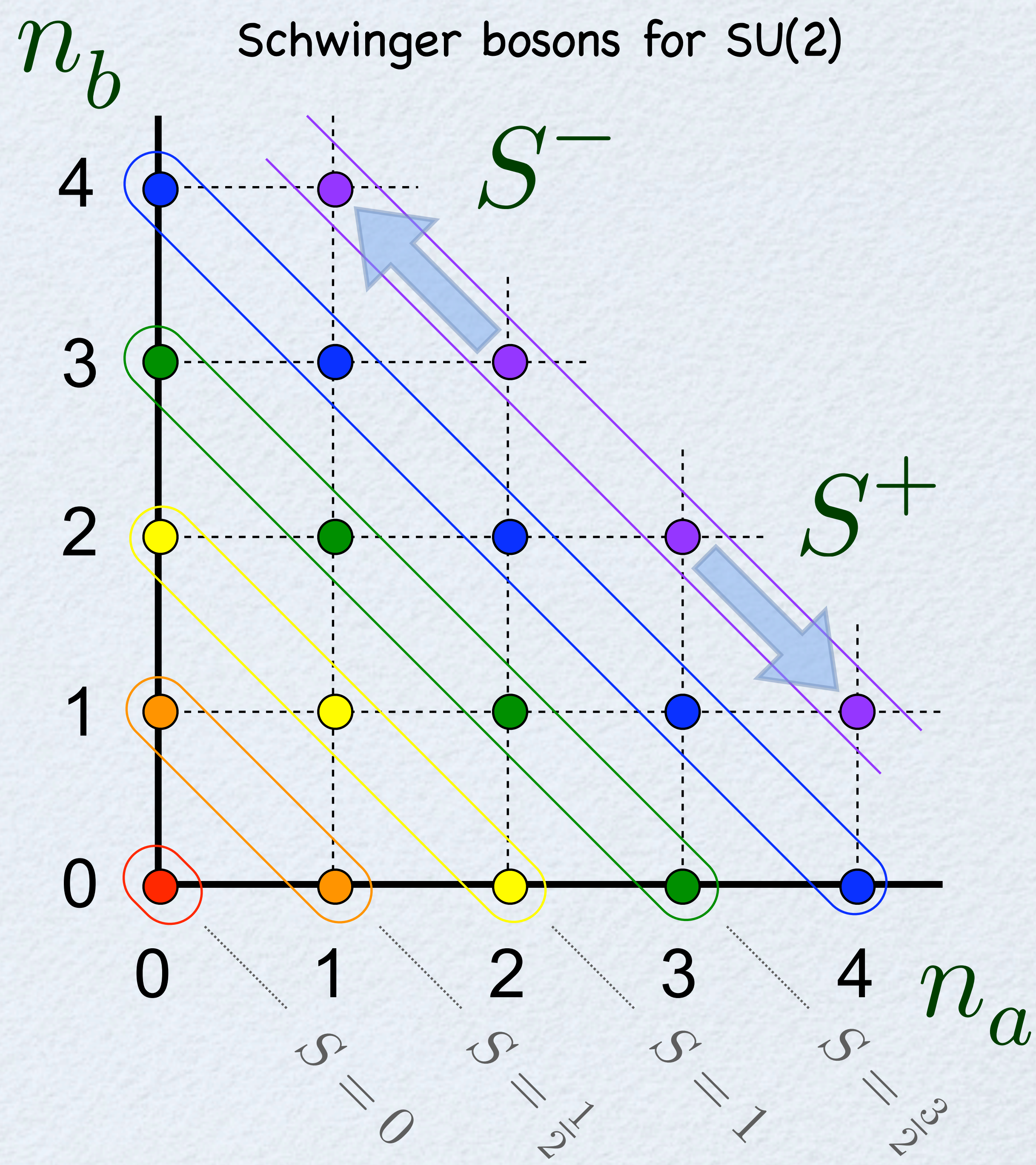
$$\boldsymbol{S}_i \cdot \boldsymbol{S}_j = S^2 - \frac{1}{2}\mathcal{A}_{ij}^\dagger \mathcal{A}_{ij}$$

$$\mathcal{A}_{ij} = a_i\, b_j - b_i\, a_j$$

$\downarrow$ N copies

$$\mathcal{A}_{ij} = \sum_{m=1}^{N} (a_{im}\, b_{jm} - b_{im}\, a_{jm})$$

$$n_c = \sum_{m=1}^{N} (a_{im}^\dagger a_{im} + b_{im}^\dagger b_{im}) \equiv \kappa N$$



which is the Sp(N) extension

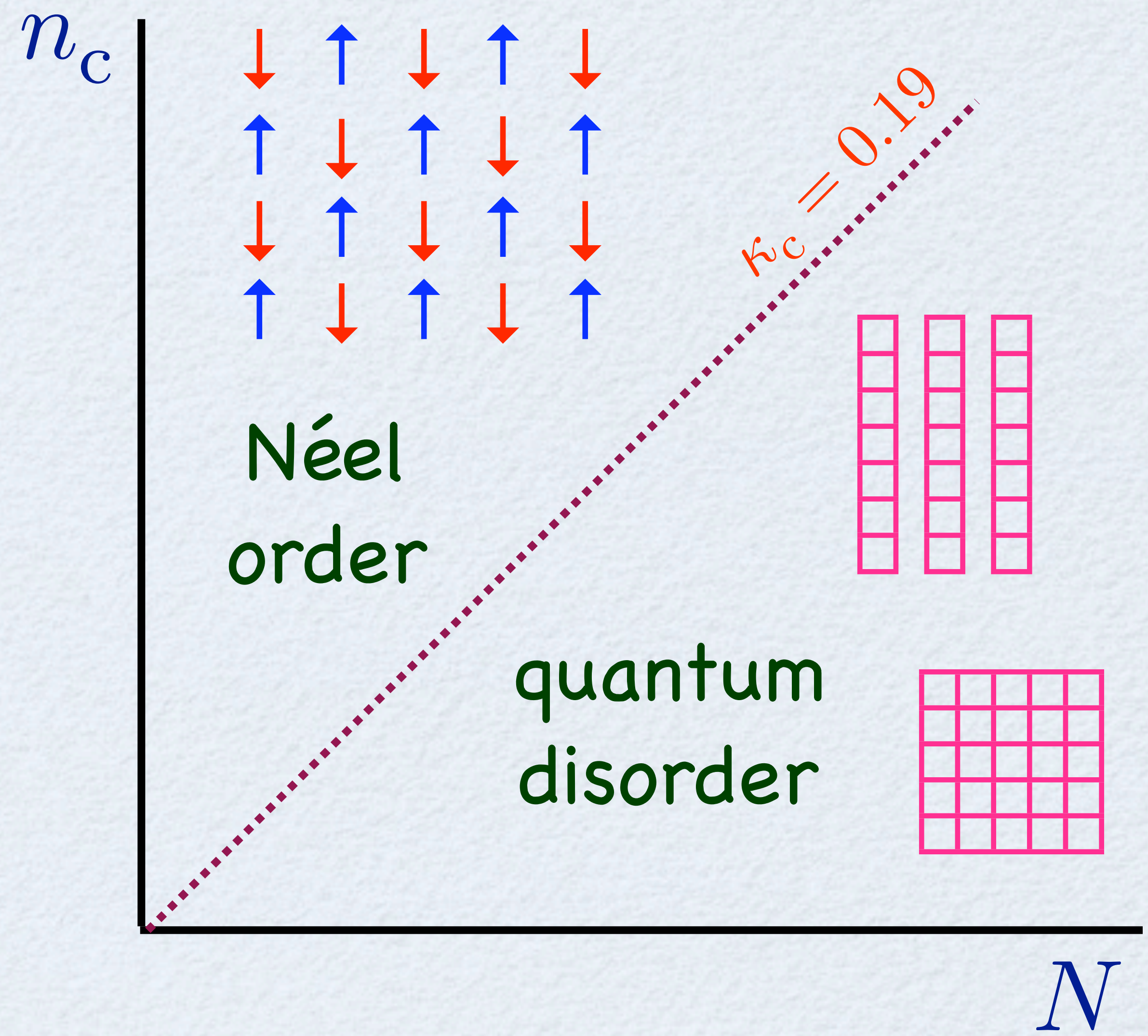
$$\mathrm{SU}(2) \cong \mathrm{Sp}(1) \quad \Rightarrow \quad \kappa = S$$

The model: Sp(N) quantum antiferromagnet

$$\mathcal{A}_{ij} = \sum_{m=1}^{N} \left(a_{im}\, b_{jm} - b_{im}\, a_{jm} \right)$$

$$n_c = \sum_{m=1}^{N} \left(a^\dagger_{im} a_{im} + b^\dagger_{im} b_{im} \right) \equiv \kappa N$$

$$\mathcal{H} = -\frac{1}{2N} \sum_{i<j} J_{ij}\, \mathcal{A}^\dagger_{ij}\, \mathcal{A}_{ij}$$

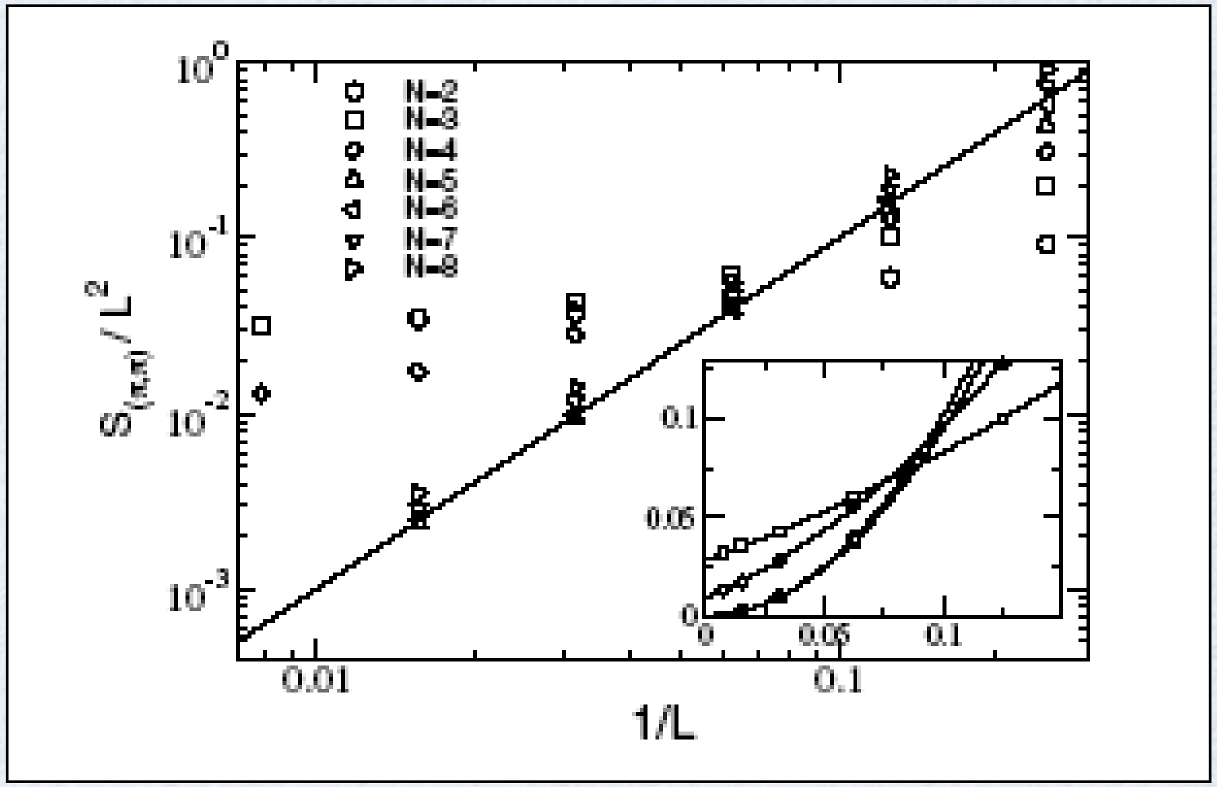


K. Harada, N. Nawashima, and M. Troyer (2003):

SU(N) antiferromagnet with n_c=1 (square lattice) via quantum Monte Carlo method.

N ≤ 4 : Néel order

N ≥ 5 : quantum disorder (columnar valence bond crystal)



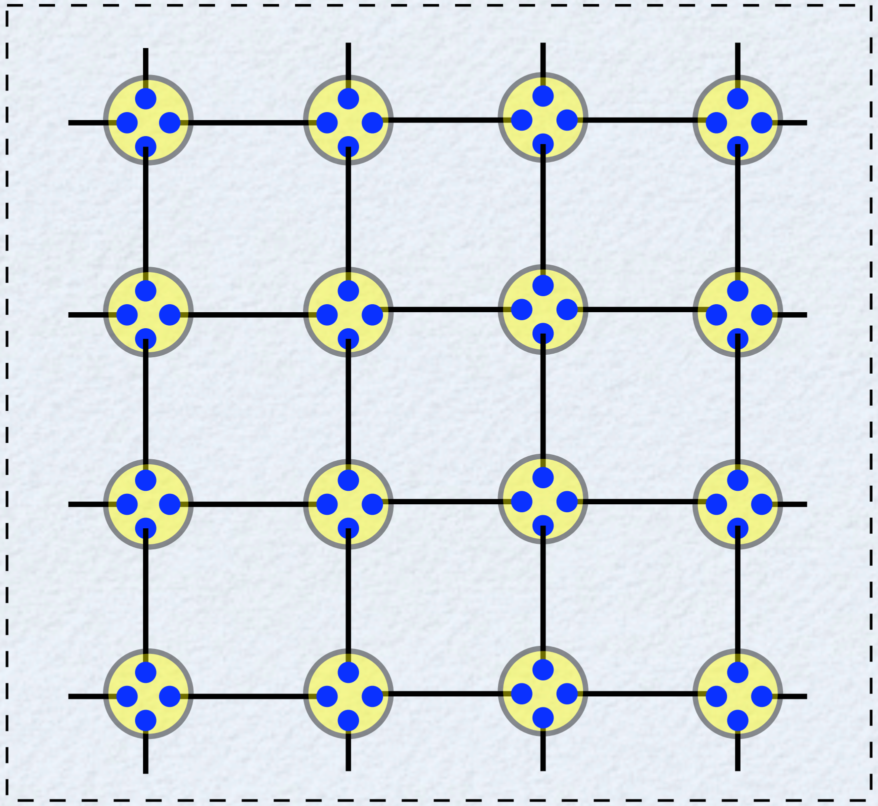
General VBS states

(AKLT, 1987; DPA, Auerbach, Haldane, 1988)

$$\big|\Psi\big\rangle = \prod_{\langle ij \rangle} (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)^{m_{ij}} \, \big|0\big\rangle$$

Schwinger boson representation
(local symmetrization is automatic)

Local spin quantum number : $S_i = \frac{1}{2} \sum_j{}' m_{ij}$



$S = 2$

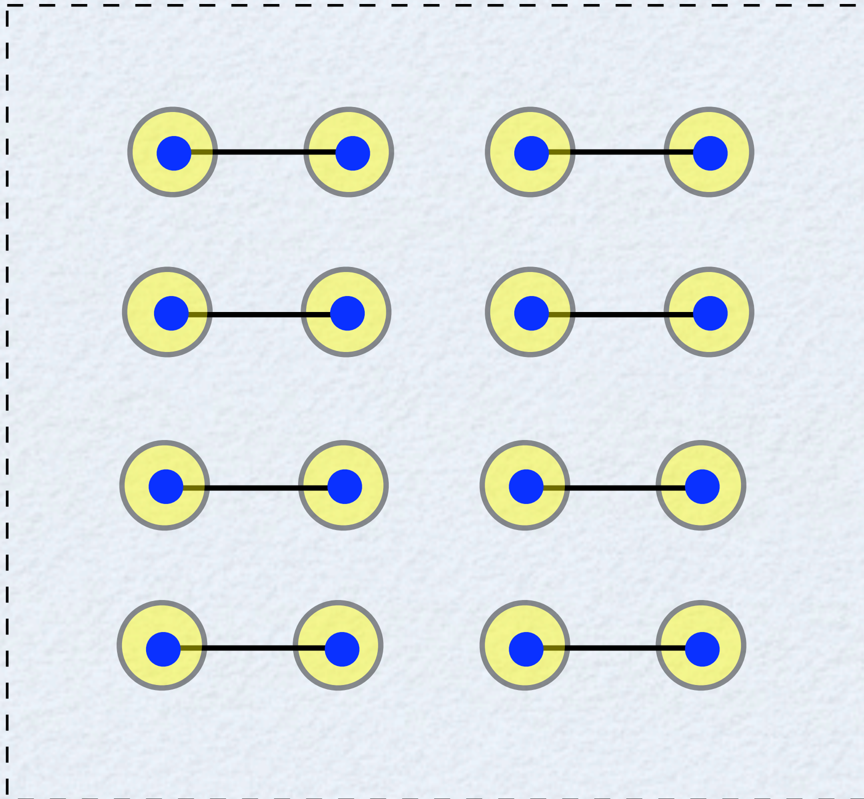
With m bond singlet operators per link, the maximum bond spin is $J_{\rm max} = 2S - m$

Then with $H = \sum_{\langle ij \rangle} \sum_{J_{\rm max}+1}^{2S} V_J \, {\rm P}_J^{[S]}(ij)$,

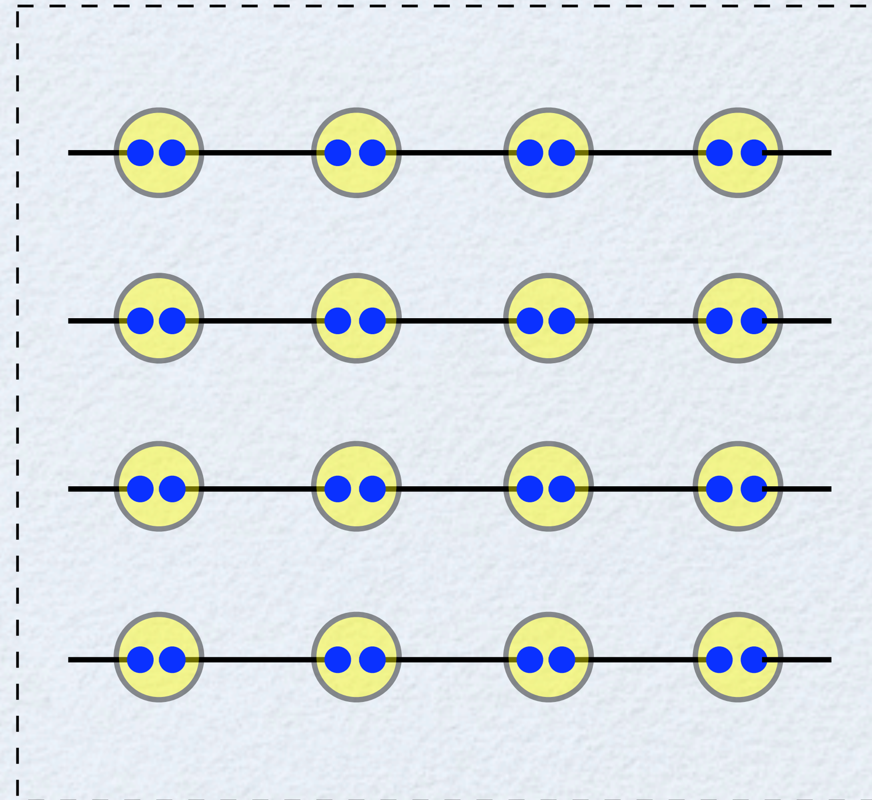
we have $H \, \big|\Psi(\mathcal{L}, m)\big\rangle = 0$.

VBS states are zero energy eigenstates.

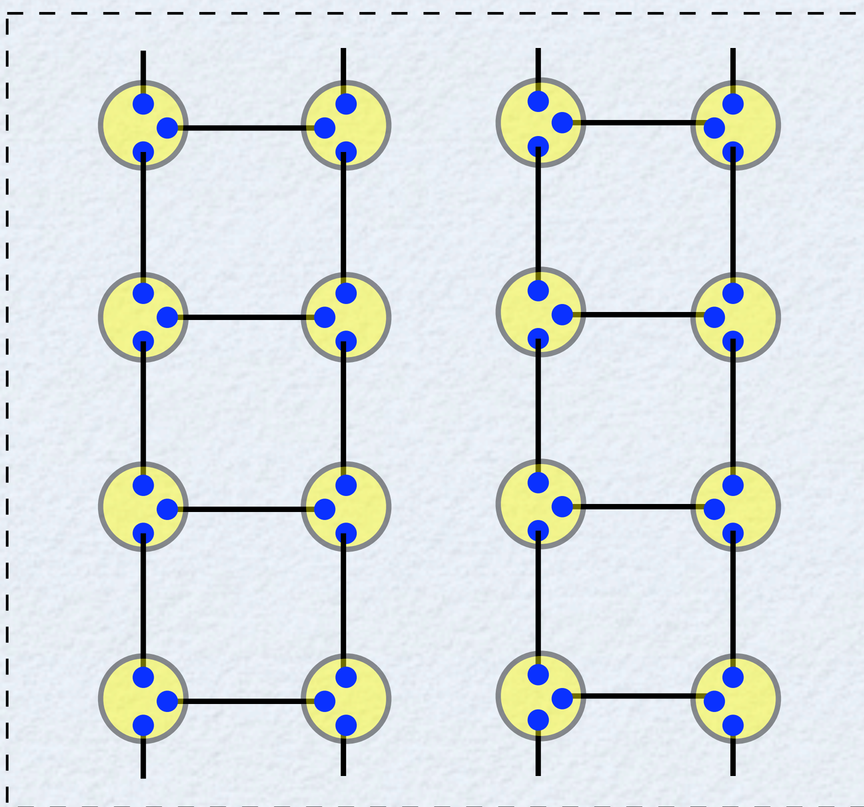
Haldane predicted ground state degeneracies in quantum disordered phases depending on 2S mod 4. These are realized with VBC states.



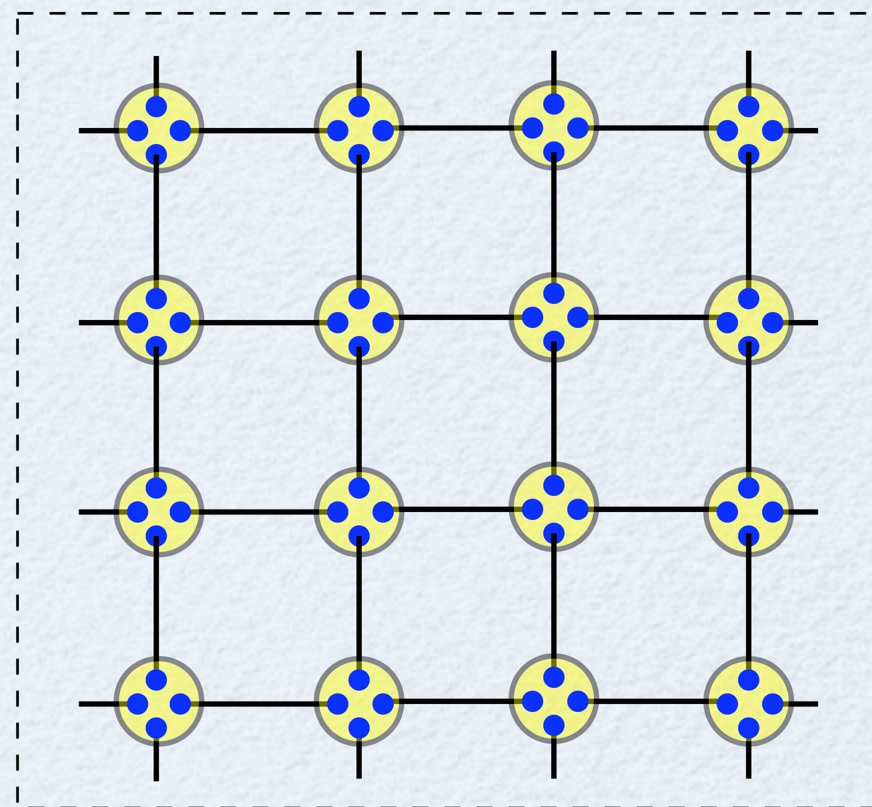
$S = \frac{1}{2}$



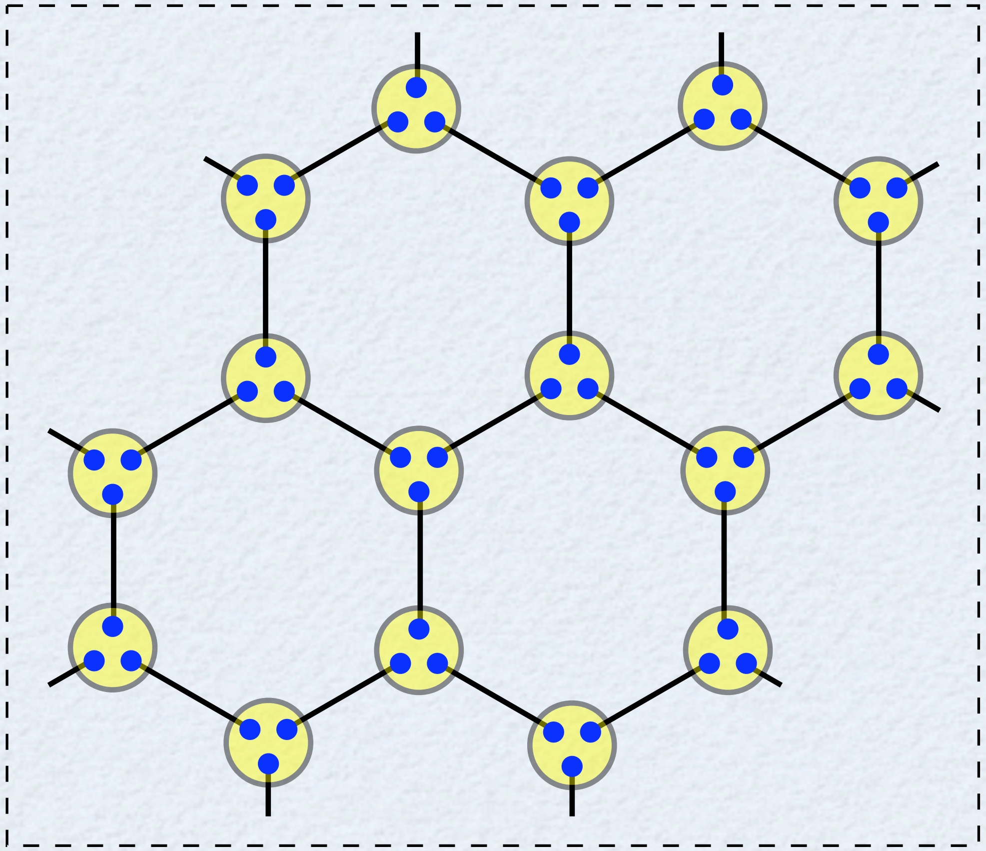
$S = 1$



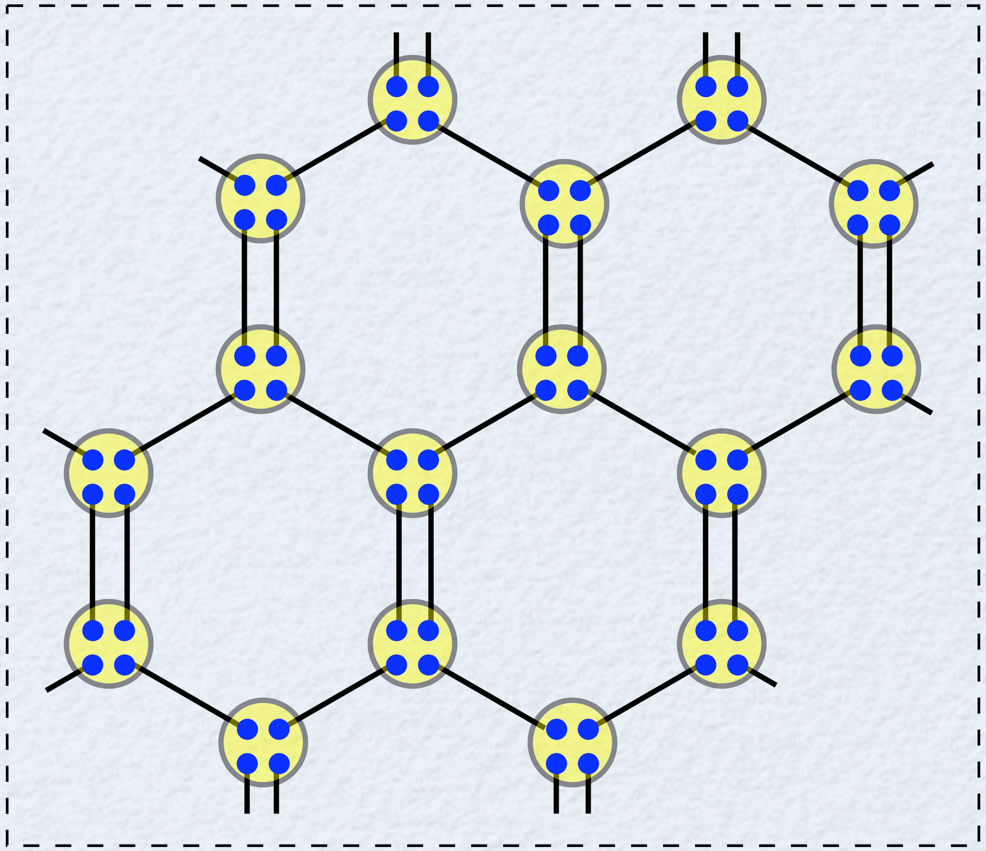
$S = \frac{3}{2}$



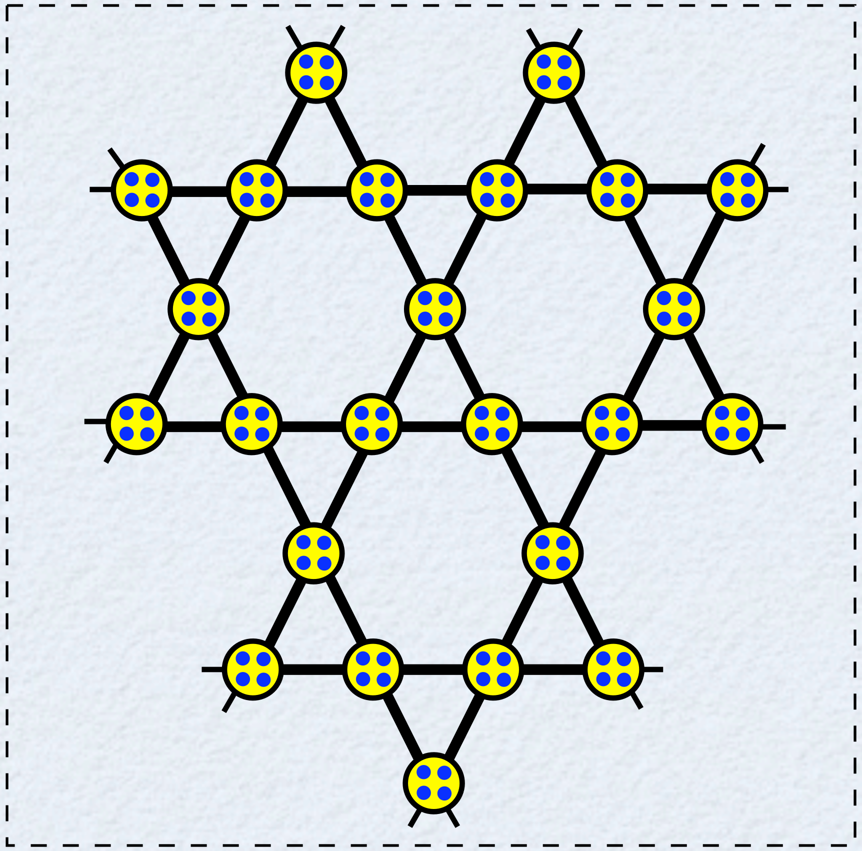
$S = 2$



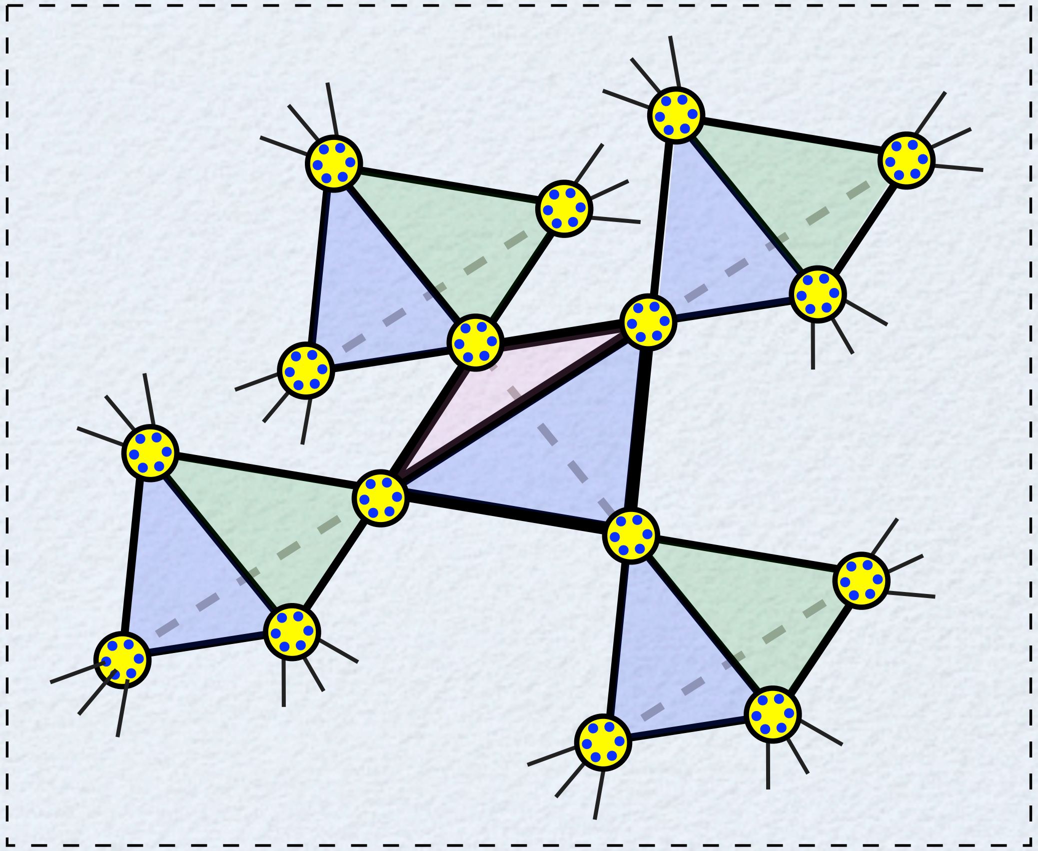
S = 3/2 honeycomb



S = 2 honeycomb

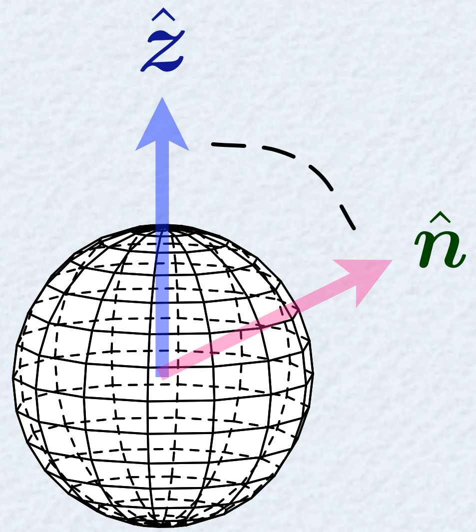


S = 2 Kagomé

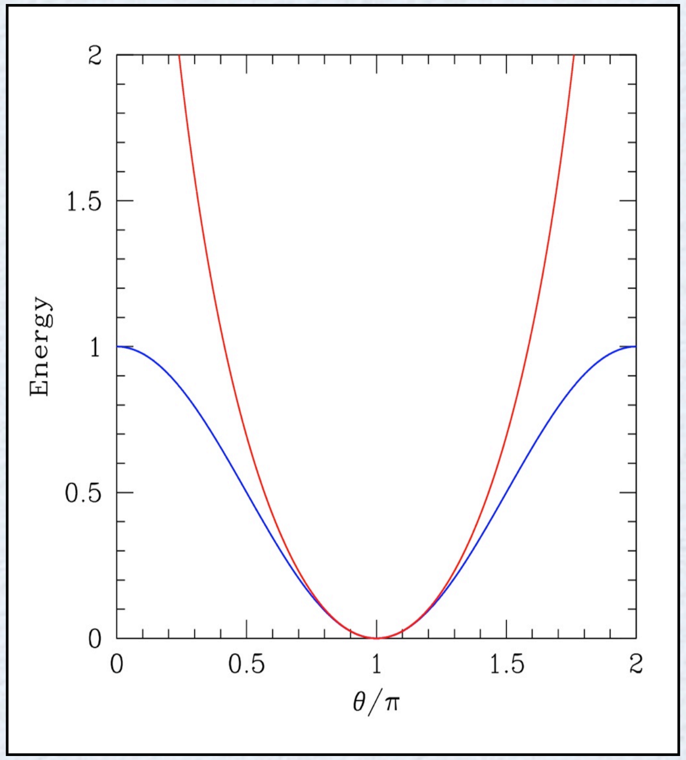


S = 3 pyrochlore

Spin coherent states



$$(\hat{\boldsymbol{n}}\cdot\boldsymbol{S})\,|\hat{\boldsymbol{n}}\rangle = S\,|\hat{\boldsymbol{n}}\rangle$$



$$|\hat{\boldsymbol{n}}\rangle = (2S\,!)^{-1}\big(ua^\dagger + vb^\dagger\big)^{2S}|0\rangle$$

$$u = \cos\left(\tfrac{1}{2}\theta\right)$$
$$v = \sin\left(\tfrac{1}{2}\theta\right)e^{i\phi}$$

matrix elements : $\langle\Phi|\hat{T}|\Psi\rangle = \int\frac{d\hat{\boldsymbol{n}}}{4\pi}\,\Phi^*(\hat{\boldsymbol{n}})\,T(\hat{\boldsymbol{n}})\,\Psi(\hat{\boldsymbol{n}})$

$$\Psi_{\rm AKLT}[\hat{\boldsymbol{n}}] = \prod_{\langle ij\rangle}(u_i v_j - v_i u_j)^m$$

$$\left|\Psi_{\rm AKLT}\right|^2 = \prod_{\langle ij\rangle}\left(\frac{1-\hat{\boldsymbol{n}}_i\cdot\hat{\boldsymbol{n}}_j}{2}\right)^m \equiv e^{-H_{\rm cl}/T}$$

classical model : $H_{\rm cl} = -\sum_{\langle ij\rangle}\ln\sin^2\left(\tfrac{1}{2}\vartheta_{ij}\right)$

temperature : $T = \frac{1}{m}$

All equal time quantum correlations in the AKLT states may be computed as the **finite temperature** ($T=m^{-1}$) correlations of a related **classical** model $H_{\rm cl}$ on the **same lattice**

VBS as a Matrix Product State

Matrix product states (d=1) :

Klümper, Schadschneider, Zittarz, 1991
Fannes, Nachtergaele, Werner, 1992
Rommer and Ostlund, 1995
Verstraete, Porras, Cirac 2004

$$\big|\Psi\big\rangle = \sum_{\{\sigma_n\}} \mathrm{Tr}\left(A^{\sigma_1} A^{\sigma_2} \cdots A^{\sigma_N}\right) \big|\sigma_1\,\sigma_2\,\cdots\,\sigma_N\big\rangle$$

Local Hilbert space : $\big|\sigma\big\rangle$, $\sigma \in \{1, \ldots, d_{\mathrm{L}})$, $A \in \mathrm{GL}(n, \mathrm{C})$

S=1 VBS chain : $\Psi[Z] = \prod_n \epsilon^{\mu\nu} z_{n,\mu}\, z_{n+1,\nu}$

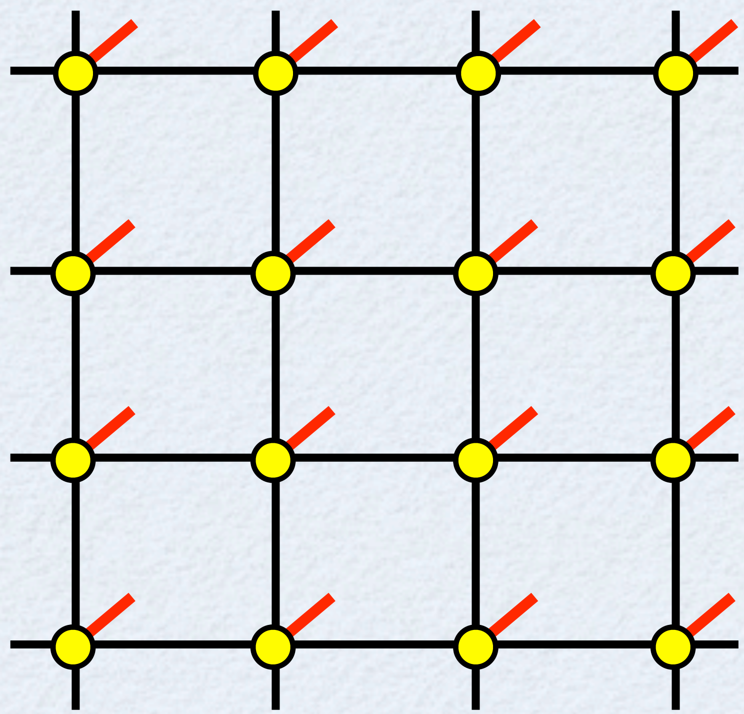
$$= \cdots \left(z_{n-1,\rho}\, \epsilon^{\mu\nu}\, z_{n-1,\mu}\right)\left(z_{n,\nu}\, \epsilon^{\alpha\beta}\, z_{n,\alpha}\right)\left(z_{n+1,\beta}\, \epsilon^{\sigma\tau}\, z_{n+1,\sigma}\right) \cdots$$

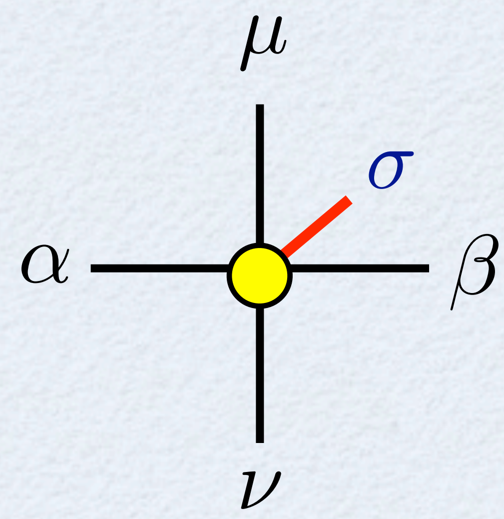
$$= \mathrm{Tr}\left[M(z_1) M(z_2) \cdots M(z_N)\right]$$

with $M_{\mu\nu}(z) = z_\mu\, \tilde{z}_\nu$, where $z = \begin{pmatrix} u \\ v \end{pmatrix}$, $\tilde{z} = \begin{pmatrix} -v \\ u \end{pmatrix}$

In the discrete basis, $A^{\sigma}_{\mu\nu} = \mathrm{L}^{\sigma}_{\mu}\, \mathrm{R}^{\sigma}_{\nu}$, a restricted class of MPS

General VBS as a Tensor Product State





is a tensor $A^{\sigma}_{\alpha\beta\mu\nu}$

$$|\Psi_{\text{TPS}}\rangle = \sum_{\{\sigma_i\}} \mathcal{C}\Big(\prod_i A^{\sigma_i}\Big)|\sigma_1, \ldots, \sigma_N\rangle$$

contract tensor indices on links

For the general VBS state, $A^{\sigma}_{\alpha\beta\mu\nu} = \mathrm{N}^{\sigma}_{\mu}\,\mathrm{S}^{\sigma}_{\nu}\,\mathrm{W}^{\sigma}_{\alpha}\,\mathrm{E}^{\sigma}_{\beta}$

The contractions are now trivial, e.g. $\mathrm{N}^{\sigma_{m,n}}_{\mu}\,\mathrm{S}^{\sigma_{m,n+1}}_{\mu} \equiv e^{-\phi(\sigma_{m,n}\,,\,\sigma_{m,n+1})}$

Correlations are those of an associated classical model :

$$\left|\Psi_{\sigma_1,\ldots,\sigma_N}\right|^2 = \exp\Big(-\sum_{\langle ij\rangle} V(\sigma_i,\sigma_j)\Big)$$

VBS phase transition

Parameswaran, Sondhi, DPA
arXiv cond-mat/0807.3189

Recall $H_{\rm cl} = -\sum_{\langle ij\rangle} \ln\left(\frac{1-\hat{\boldsymbol{n}}_i\cdot\hat{\boldsymbol{n}}_j}{2}\right)$ with $T = \frac{1}{m}$

Mean field theory : $\hat{\boldsymbol{n}}_i = \boldsymbol{m}_i + \delta\hat{\boldsymbol{n}}_i$, $\langle\hat{\boldsymbol{n}}_i\rangle = \boldsymbol{m}_i$, $\hat{\boldsymbol{h}}_i = -{\sum_j}' \frac{\boldsymbol{m}_j}{1-\boldsymbol{m}_i\cdot\boldsymbol{m}_j}$

MF Hamiltonian : $H^{\rm MF}_{\rm cl} = -\sum_i \boldsymbol{h}_i\cdot\hat{\boldsymbol{n}}_i$, self-consistency : $m_i = \coth\left(\frac{h_i}{T}\right) - \frac{T}{h_i}$

Assuming a uniform solution on a bipartite lattice, $h=zm/(1+m^2)$, and

$$T_{\rm c}^{\rm MF} = \frac{z}{3} \quad \Leftrightarrow \quad m_{\rm c}^{\rm MF} = \frac{3}{z}$$

VBS states with $m>m_c$ will have Néel order, hence **if $m_c<1$ ($T_c>1$) on a given lattice, all VBS states on that lattice will be ordered**

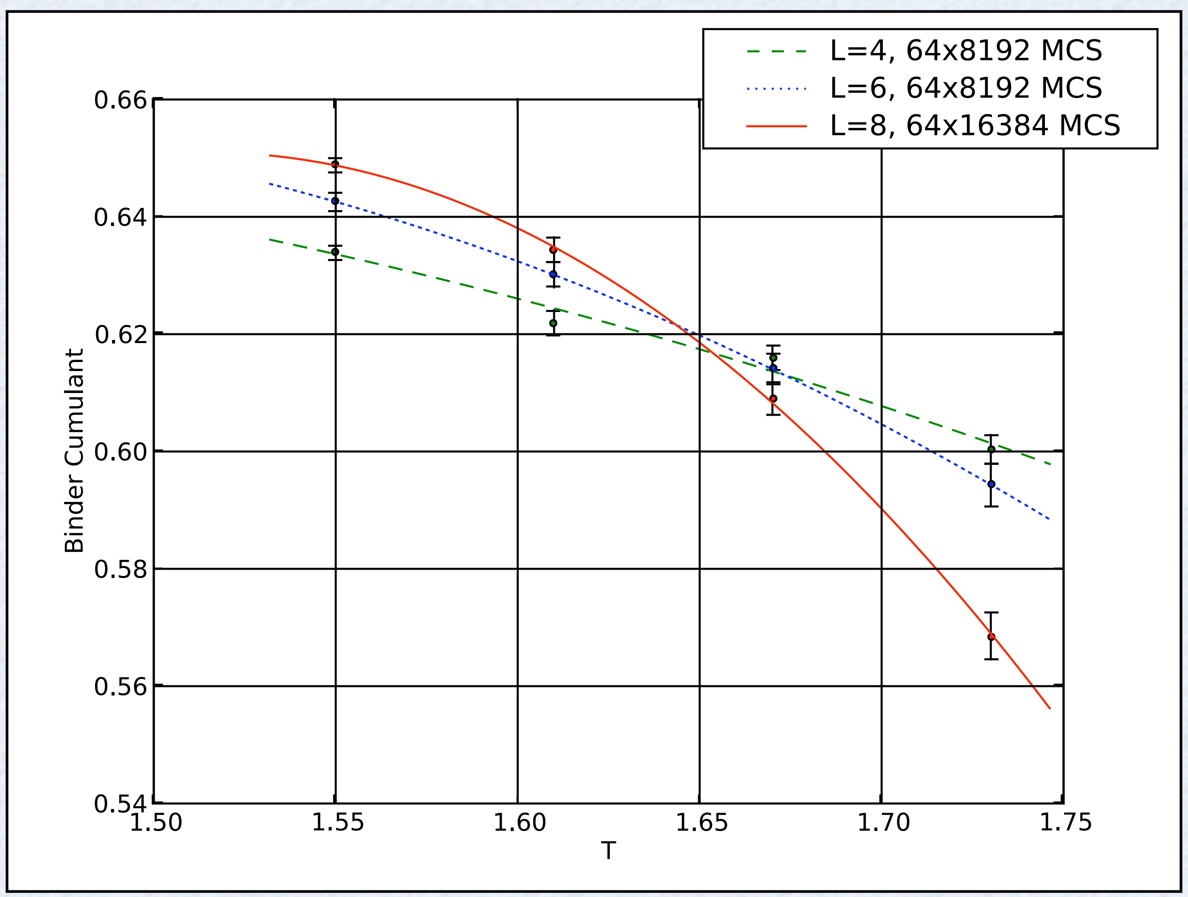
Since all lattices in $d>2$ dimensions have $z>3$, MFT predicts no quantum disorder

However, MFT overestimates T_c, hence underestimates m_c - there may be hope!

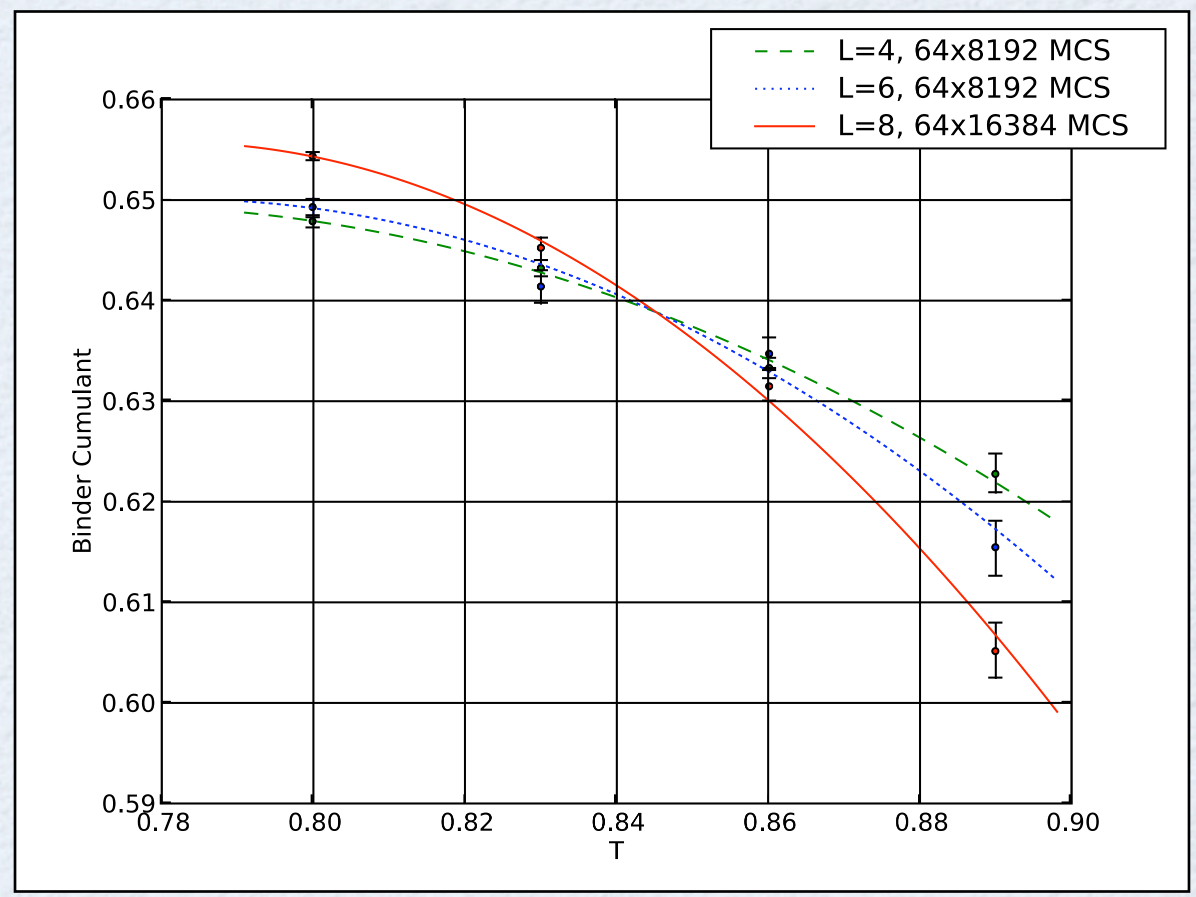
The phase transition can be investigated using classical Monte Carlo

- Simple single-spin Metropolis algorithm
- 'Multithreaded' Monte Carlo: run M Markov chains with N sites each; reject first N/2 sites and use the rest to average
- Unbiased error estimate from standard deviation of the M independent averages
- Test using high temperature expansion, T_c for known models; MFT gives $T_c^{MF} = z/3$
- Estimate T_c using Binder cumulant crossing $B = 1 - \frac{\langle (\boldsymbol{M}^2)^2 \rangle}{3\langle \boldsymbol{M}^2 \rangle^2}$, $\boldsymbol{M} = \sum_i \eta_i \hat{\boldsymbol{n}}_i$

simple cubic lattice : T_c=1.64 (T_c^{MF}= 2)

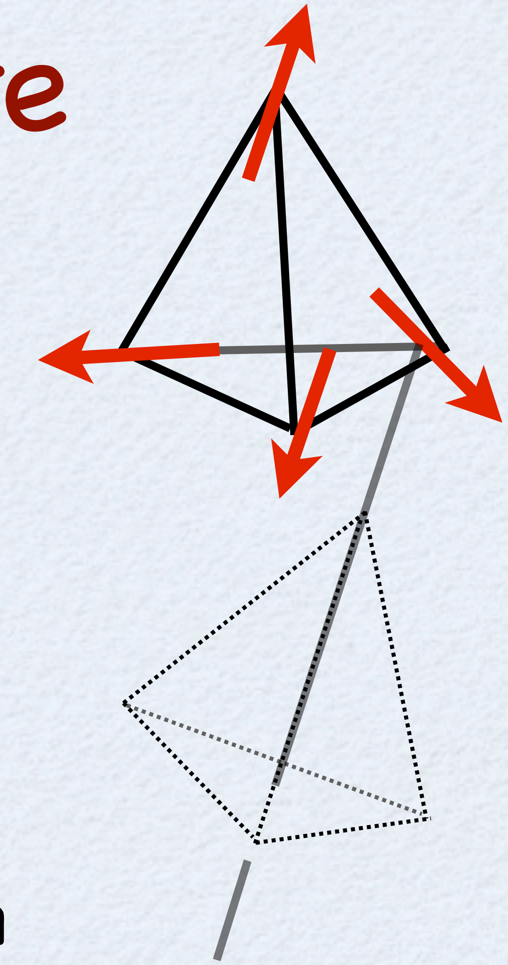


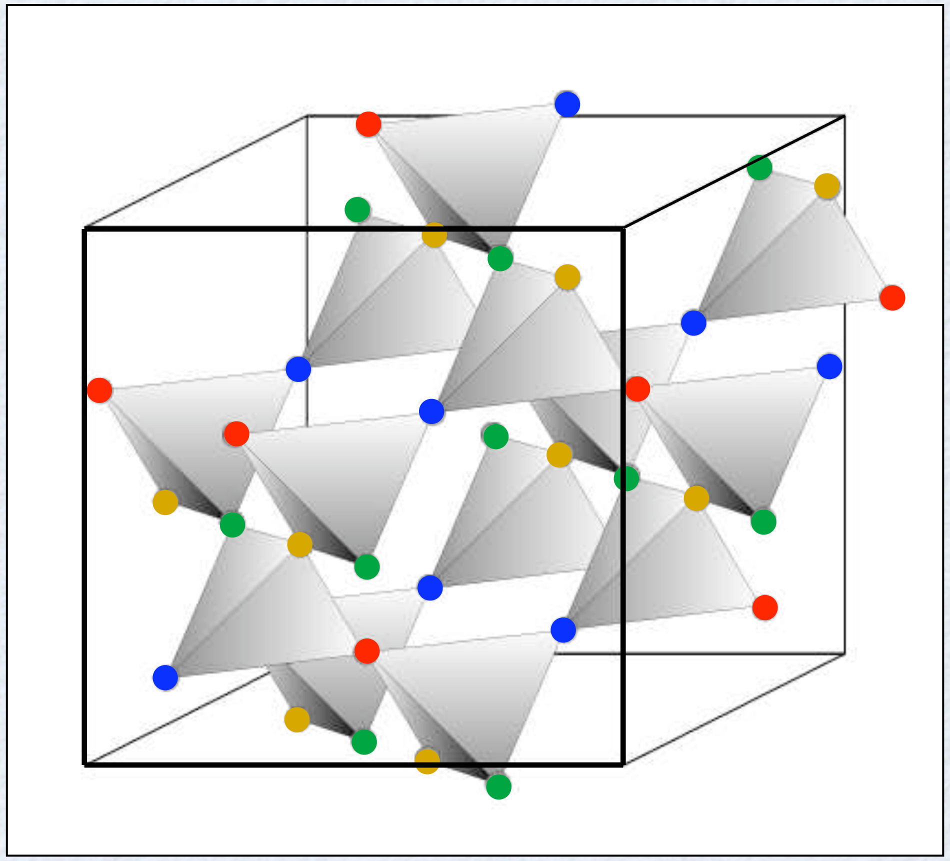
diamond lattice : T_c=0.834 (T_c^{MF}=1.33)



Pyrochlore VBS state

Parameswaran, Sondhi, DPA, Moessner

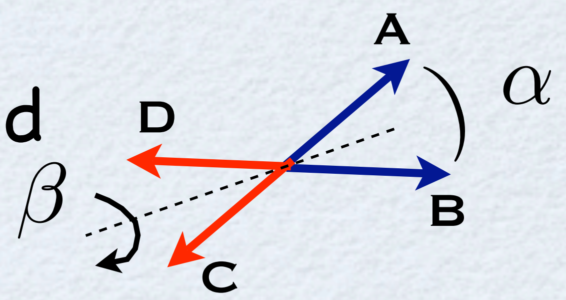




- z=6 ☞ S = 3m

- highly frustrated lattice, with corner-sharing tetrahedra and kagomé planes

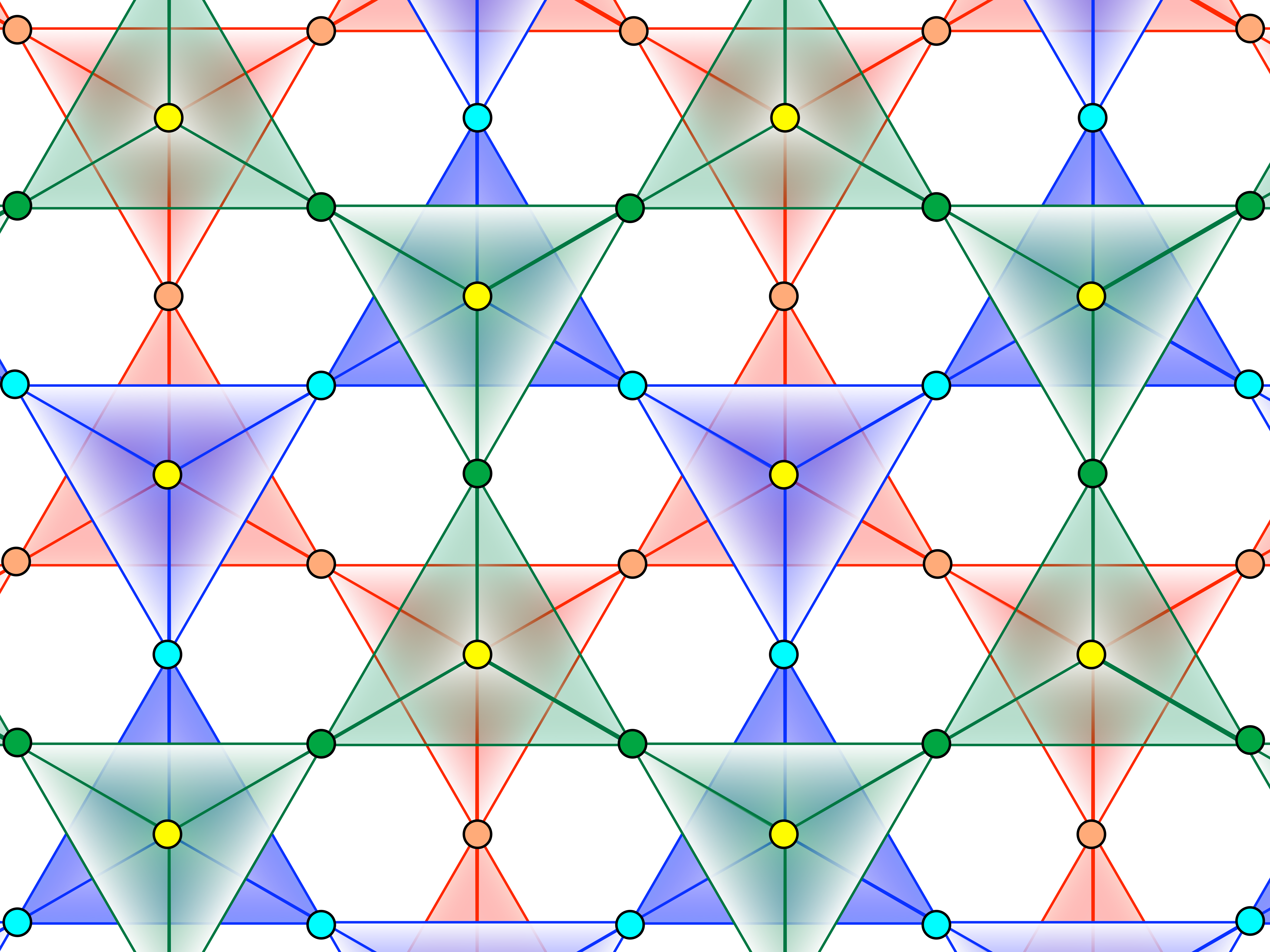
- Heisenberg AFM on tetrahedron $(\mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3 + \mathbf{S}_4)^2$ has **five**-dimensional ground state manifold (two DOF plus global O(3) rotations)



- Large-N analysis of pyrochlore Heisenberg AFM (Isakov et al., 2004) finds system is paramagnetic down to T = 0

- AKLT H_{cl} has unique single tetrahedron ground state up to global O(3), no local zero modes -- "order by disorder" at sufficiently low T ???

- Topology of ground state manifold not simply connected



Simplex Solids

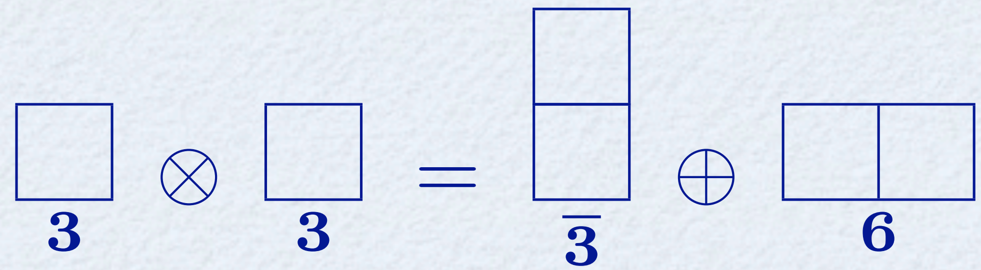
DPA, 2008

- generalization of VBS states to SU(N)

- For SU(2), we can always create a singlet with two spin-S objects :

$$S \otimes S = 0 \oplus 1 \oplus 2 \oplus \cdots \oplus 2S$$

This is no longer the case if SU(2) is replaced with SU(N).
E.g. for SU(3),



- Two ways to make singlets from SU(N) :

1. On bipartite lattices, use conjugate representation on B sublattice:

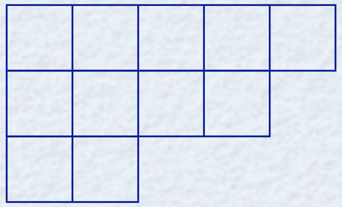
$$N \otimes \overline{N} = \underset{1}{\bullet} \oplus \underset{N^2-1}{\mathrm{adj}}$$

(this is the scheme originally used in the large-N `Schwinger boson' MFT)

2. Retain same representation, but extend singlet over N lattice sites:

$$\left|\Gamma\right\rangle = \epsilon^{\alpha_1 \cdots \alpha_N}\, b^\dagger_{\alpha_1}(i_1) \cdots b^\dagger_{\alpha_N}(i_N)\left|0\right\rangle$$

For N = 3, this is the color singlet from QCD.

Representations of SU(N) classified by Young tableaux : 

We will assume each site in (p,0) representation 

SU(N) spin operators:

$$S^\alpha_\beta = b^\dagger_\alpha\, b_\beta - \frac{p}{N}\,\delta_{\alpha\beta}$$ (N flavors of Schwinger bosons)

SU(N) algebra : $\left[S^\alpha_\beta\,,\,S^\mu_\nu\right] = \delta_{\beta\mu}\,S^\alpha_\nu - \delta_{\alpha\nu}\,S^\beta_\mu$

Define the N-simplex singlet creation operator,

$$\mathcal{R}_\Gamma^\dagger = \epsilon^{\alpha_1 \dots \alpha_N} \, b_{\alpha_1}^\dagger(\Gamma_1) \cdots b_{\alpha_N}^\dagger(\Gamma_N)$$

This is the generalization of $a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger$ for SU(2)

The general simplex solid wavefunction is then given by

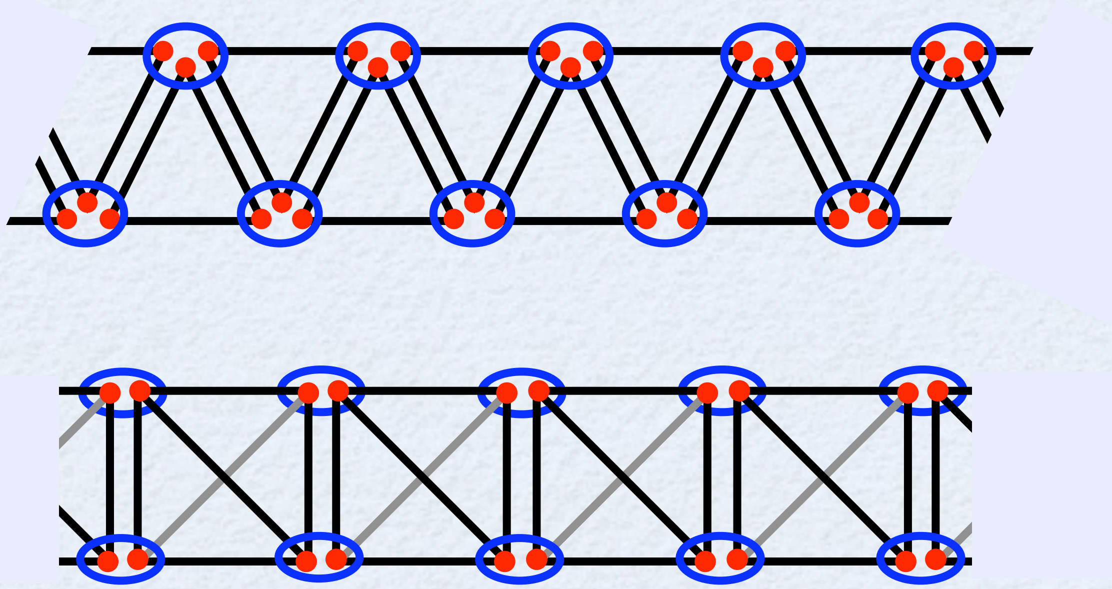
$$\left|\Psi(\mathcal{L}\,;\,M)\right\rangle = \prod_\Gamma \left(\mathcal{R}_\Gamma^\dagger\right)^M \left|0\right\rangle \quad \text{with} \quad p = \zeta M$$

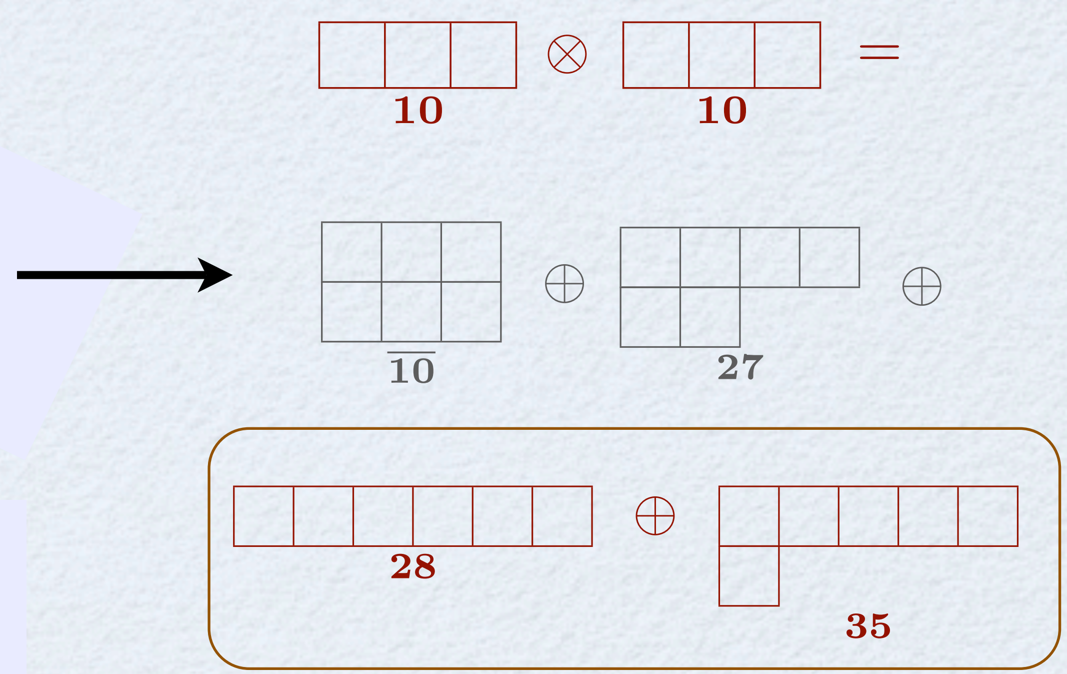
$\zeta =$ number of simplices associated with each site

Two examples in d = 1 :

SU(3)

SU(4)





missing representations

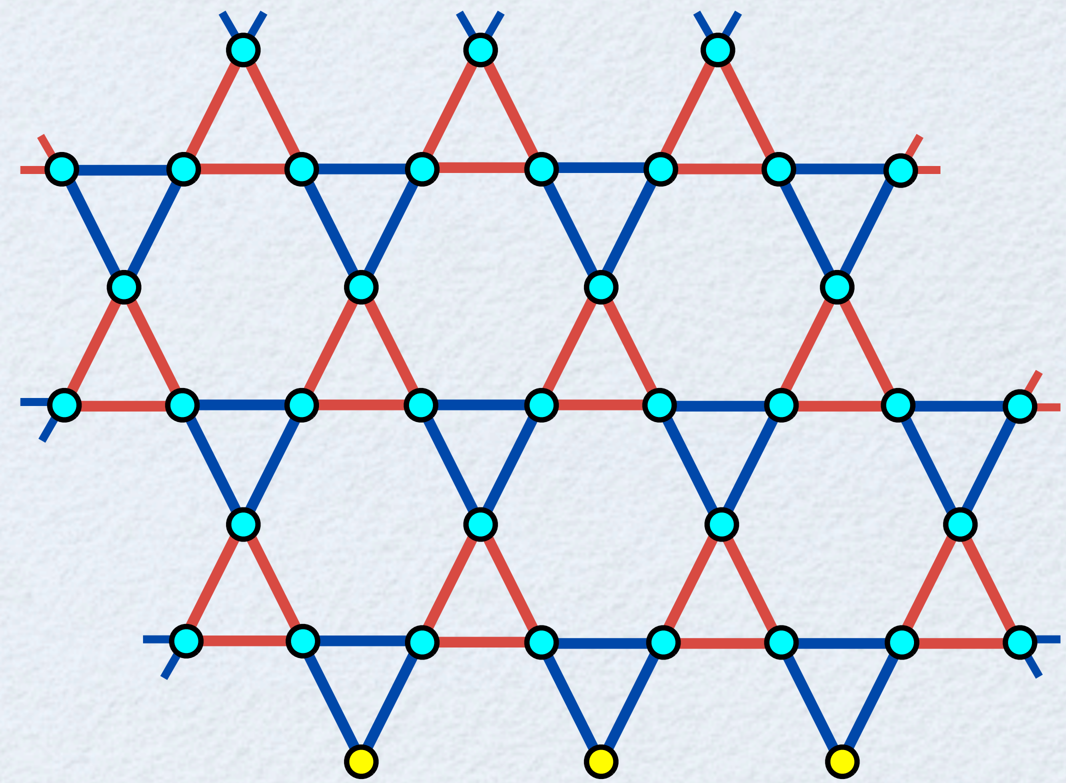
Greiter and Rachel, 2007

SU(3) fits nicely on the kagomé :

$$|\Psi\rangle = \prod_{\triangle} \mathcal{R}^{\dagger}_{\triangle} \prod_{\bigtriangledown} \mathcal{R}^{\dagger}_{\bigtriangledown} |0\rangle$$

$$\underset{\mathbf{6}}{\square\!\square} \otimes \underset{\mathbf{6}}{\square\!\square} = \underset{\bar{\mathbf{6}}}{\begin{smallmatrix}\square\square\\\square\square\end{smallmatrix}} \oplus \underset{\mathbf{15}}{\begin{smallmatrix}\square\square\square\\\square\end{smallmatrix}} \oplus \boxed{\underset{\mathbf{15}}{\square\!\square\!\square\!\square}}$$

Hamiltonian : $\mathcal{H} = \sum_{\langle ij \rangle} \mathrm{P}_{\square\square\square\square}(ij)$

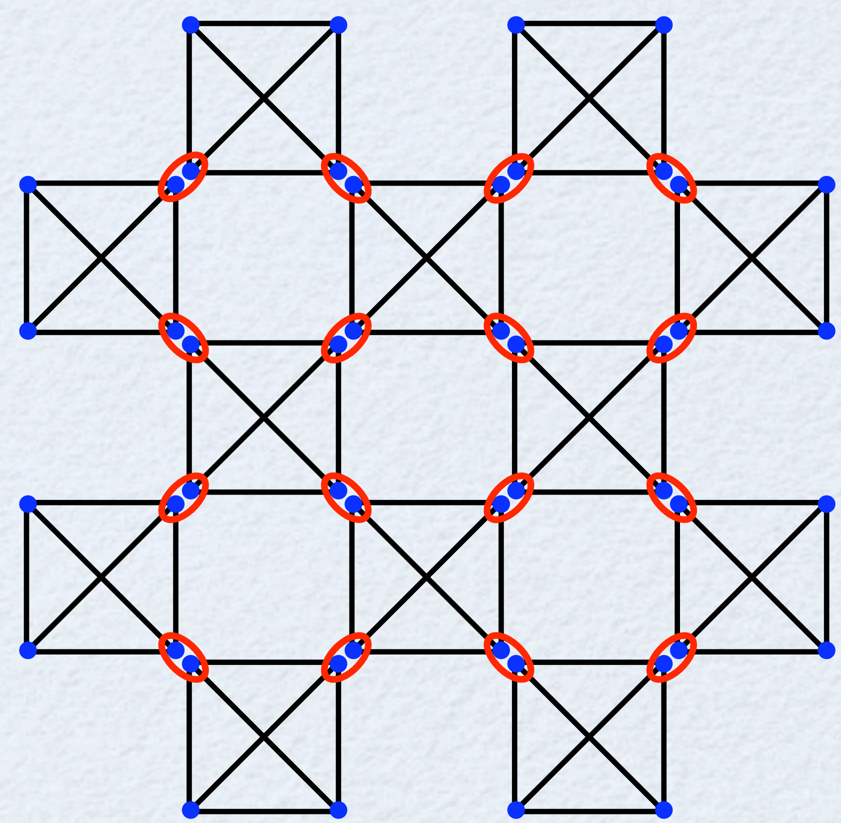


Fractionalization at the edge

Bulk spins ● in $\underset{6}{\square\!\square}$ representation

while boundary spins ● in $\underset{3}{\square}$

Recapitulates situation for AKLT states



$\underset{10}{\square\!\square}$ SU(4) state on checkerboard

Classical simplex Hamiltonian

SU(N) coherent states : $|z\,;\,p\rangle = \frac{1}{\sqrt{p!}}\Big(z_1\, b_1^\dagger + \ldots + z_N\, b_N^\dagger\Big)^p\,|0\rangle$

Find $\big|\Psi\big|^2 = \prod_\Gamma \big|R_\Gamma\big|^{2M} = e^{-H_{\rm cl}/T}$ with $H_{\rm cl} = -\sum_\Gamma \ln\big|R_\Gamma\big|^2$

where $\big|R_\Gamma\big|^2 = \epsilon^{\alpha_1\cdots\alpha_N}\,\epsilon^{\beta_1\cdots\beta_N}\,Q_{\alpha_1\beta_1}(\Gamma_1)\cdots Q_{\alpha_N\beta_N}(\Gamma_N)$ and $Q_{\alpha\beta}(i) = \bar z_\alpha(i)\, z_\beta(i)$

Mean field ansatz: $Q_{\alpha\beta}(i=\Gamma_\sigma) = \frac{1}{N}\,\delta_{\alpha\beta} + m\left({\rm P}^\sigma_{\alpha\beta} - \frac{1}{N}\,\delta_{\alpha\beta}\right) + \delta Q_{\alpha\beta}(i)$

(m: order parameter; ${\rm P}^\sigma_{\alpha\beta}$: projector)

Find $M_{\rm c}^{\rm MF}(N,\zeta) = \frac{1}{T_{\rm c}^{\rm MF}(N,\zeta)} = \frac{N^2-1}{\zeta} = \frac{\text{\# directions Q can fluctuate}}{\text{simplex coordination number}}$

The simplex solids are more likely to form quantum disordered states.

E.g. for the SU(4) pyrochlore SS, find $M_{\rm c}^{\rm MF} = \frac{15}{2}$. The pyrochlore SS is unfrustrated, but should exhibit "order by disorder".

Order by disorder

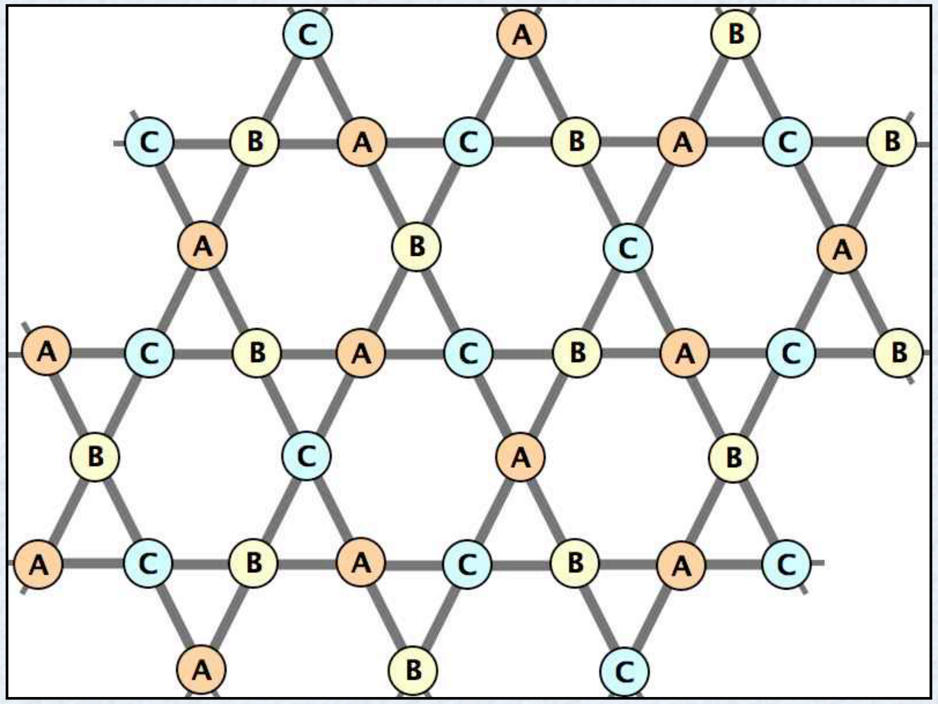
Derive a NLσM : $z(i) = \omega_{\sigma(i)} \left(1 - \pi_i^\dagger \pi_i\right)^{1/2} + \pi_i$

Hamiltonian : $H_{\rm LT} = \sum_{\langle ij \rangle} \left| \pi_i^\dagger \omega_{\sigma(j)} + \omega_{\sigma(i)}^\dagger \pi_j \right|^2 + \lambda \left(\mathcal{N} |\chi|^2 + \sum_i \pi_i^\dagger \pi_i - \mathcal{N} \right)$

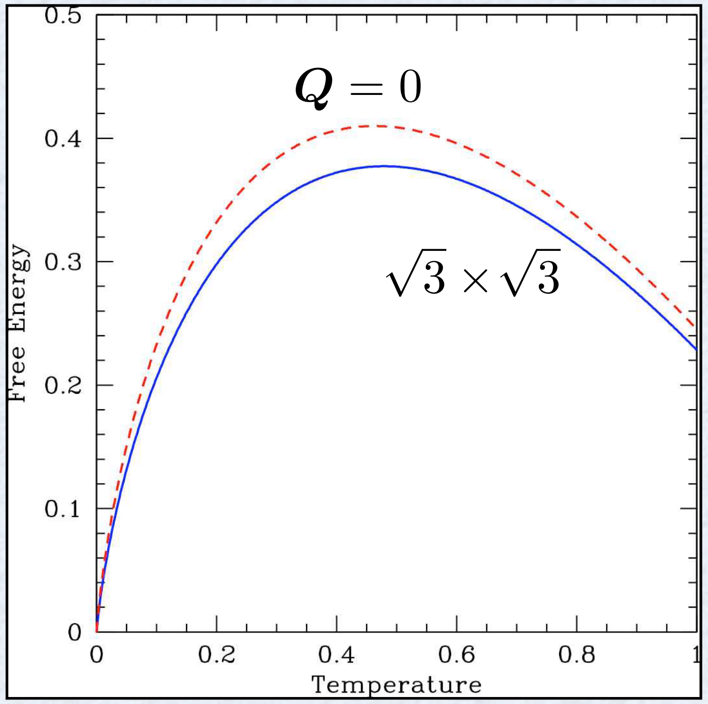
Free energy and mean field equation :

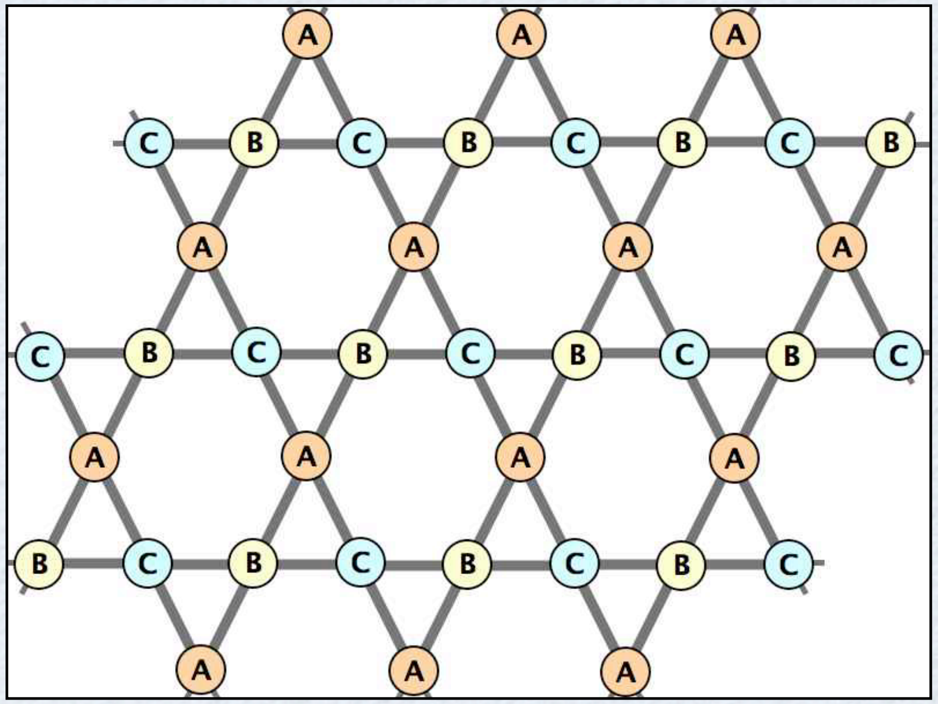
$$\frac{F}{\mathcal{N}} = -\lambda + \lambda |\chi|^2 + T \int_0^\infty d\varepsilon \, g(\varepsilon) \, \ln\left(\frac{\varepsilon + \lambda}{T}\right)$$

$$1 = |\chi|^2 + T \int_0^\infty d\varepsilon \, \frac{g(\varepsilon)}{\varepsilon + \lambda}$$



$Q = 0$ state





$\sqrt{3} \times \sqrt{3}$ state

Supersymmetric VBS states

DPA, Hasebe, Qi, Zhang (in preparation)

Parallels between VBS states and Laughlin-Haldane WF for FQHE :

$$\Psi_m^{\mathrm{LH}} = \prod_{i<j} (u_i\, v_j - v_i\, u_j)^m \qquad \Psi_m^{\mathrm{AKLT}} = \prod_{\langle ij \rangle} (u_i\, v_j - v_i\, u_j)^m$$

$$H = \sum_{i<j} \sum_{L>m} V_L\, \mathrm{P}_L(ij) \qquad H = \sum_{\langle ij \rangle} \sum_{J>2S-m} V_J\, \mathrm{P}_J(ij)$$

Supersymmetric extension (K. Hasebe, 2005) :

$$\widetilde{\Psi}_m = \prod_{i<j} (u_i\, v_j - v_i\, u_j + \theta_i\, \theta_j)^m$$

$$\{\theta_i\,,\, \theta_j\} = 0$$

Grassmann variables

Expand in powers of Grassmanns :

$$\widetilde{\Psi}_m = \Psi_m^{\mathrm{LH}} \cdot \left\{ 1 + m \sum_{i<j} \frac{\theta_i\, \theta_j}{u_i\, v_j - v_i\, u_j} + \ldots + m^{N/2}\, \theta_1 \cdots \theta_N \; \mathrm{Pf}\left(\frac{1}{u_i\, v_j - v_i\, u_j} \right) \right\}$$

SUSY VBS state:

$$\left|\Psi(\mathcal{L},M,r)\right\rangle = \prod_{\langle ij\rangle} \big(\overbrace{\epsilon_{\mu\nu}\, b^\dagger_{i\mu} b^\dagger_{j\nu} + r f^\dagger_i f^\dagger_j}^{\chi^\dagger_{ij}}\big)^M \left|0\right\rangle$$

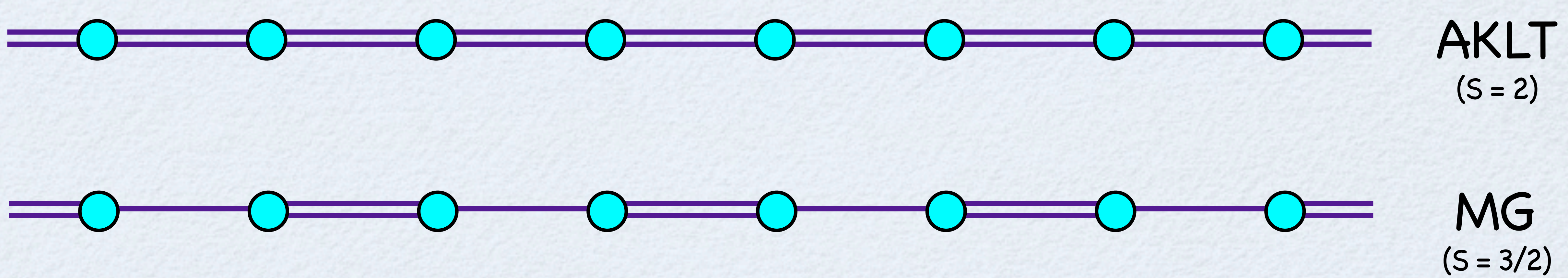
$\chi^\dagger_{ij}$ creates a linear combination of spin singlet and hole pair on link (ij).

It transforms as a singlet under the superalgebra OSp(1|2). "SVBS" state.

The c-number r can interpolate between two limits :

$$\left|\Psi(\mathcal{L},M,r)\right\rangle = \begin{cases} \left|\Psi_{\text{AKLT}}(\mathcal{L},M)\right\rangle & \text{if } r=0 \\ \left|\Psi_{\text{RVB}}(\mathcal{L},M-1)\right\rangle & \text{if } r=\infty \end{cases}$$

In d = 1, the RVB state is the Majumdar-Ghosh state :



In d > 1, the r → ∞ limit is an RVB state.

Sketch of basic connections :

AKLT (S = 1 chain) ⟷ MG (S = 1/2 chain)

↕ ↕

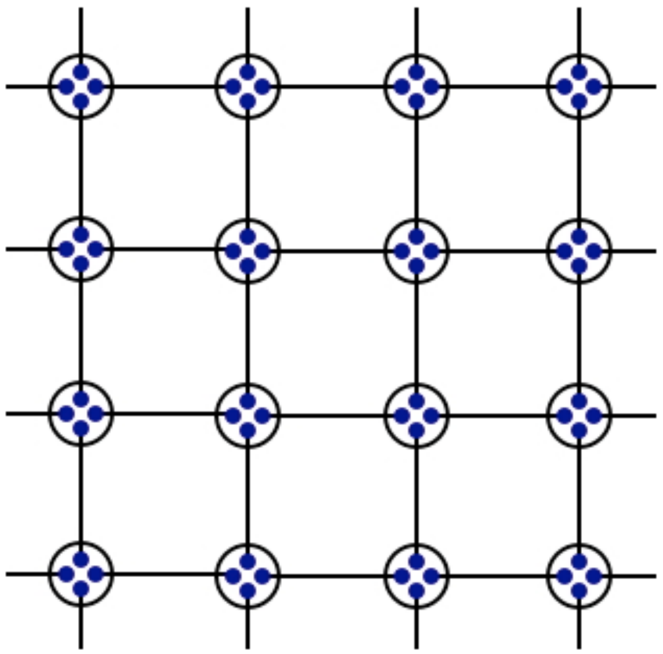
AKLT (S = 2 square) ⟷ RVB (S = 3/2 square)

↕ ↕

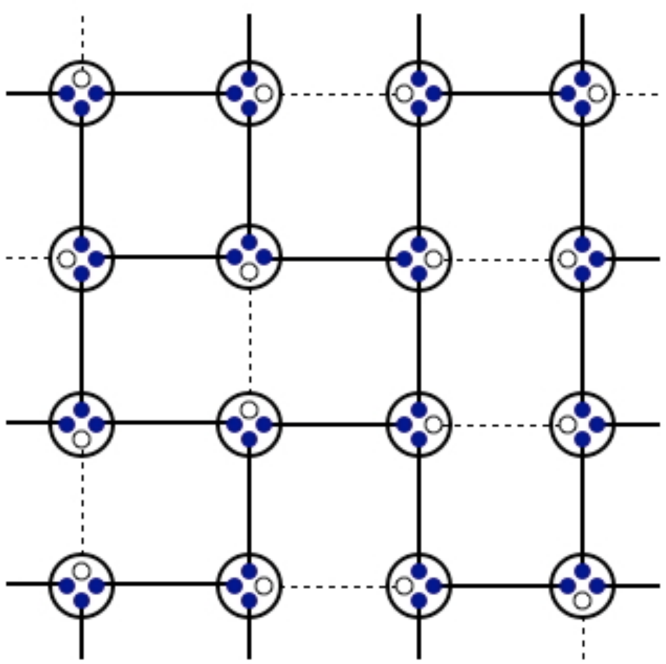
Laughlin (ν = ½ bosons) ⟷ Pfaffian (ν = ½ fermions)

↕

Haffnian (2 fermion species)



AKLT $(S = 2)$



RVB $(S = \frac{3}{2})$

OSp(1|2)

2p+1 states in supermultiplet $b^\dagger_\uparrow b_\uparrow + b^\dagger_\downarrow b_\downarrow + f^\dagger f = p$

Generators (5):

$$L_a = \tfrac{1}{2}\, b^\dagger_\mu\, \sigma^a_{\mu\nu}\, b_\nu$$

$$K_\sigma = \tfrac{1}{2}\big(x^{-1} f b^\dagger_\sigma + \sigma\, x f^\dagger b_{-\sigma}\big)$$

Algebra:

$$\{K_\mu\,,\,K_\nu\} = \tfrac{1}{2}\big(i\sigma^y\sigma^a\big)_{\mu\nu}\, L_a$$

$$\big[L_a\,,\,K_\mu\big] = \tfrac{1}{2}\sigma^a_{\nu\mu}\, K_\nu$$

$$\big[L_a\,,\,L_b\big] = i\epsilon_{abc}\, L_c$$

Casimir: $\mathcal{C} = \boldsymbol{L}^2 + \epsilon_{\mu\nu}\, K_\mu K_\nu$

SVBS chains

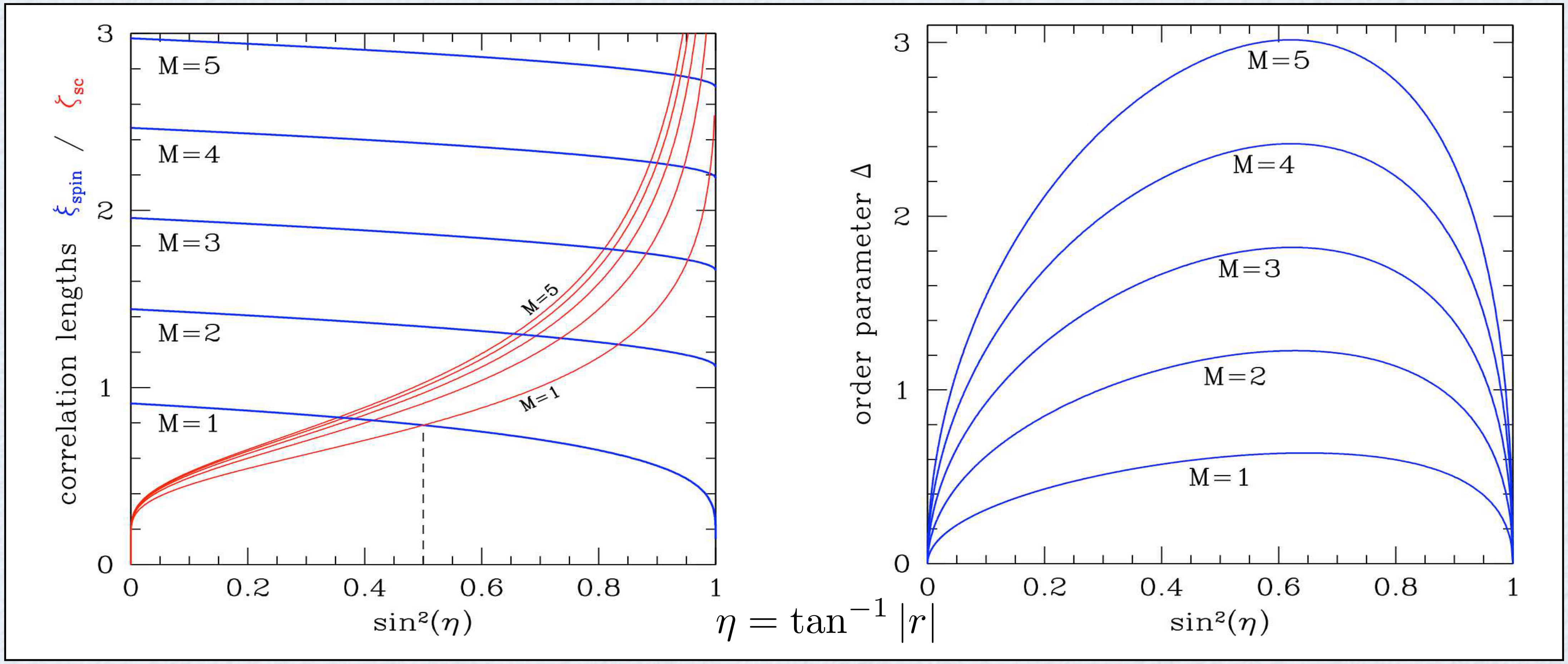
General wavefunction : $\left|\Psi\right\rangle = \prod_i \left(a_i^\dagger\, b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger + r\, f_i^\dagger\, f_{i+1}^\dagger\right)^M \left|0\right\rangle$

Correlations computed using spin-hole coherent states (Auerbach 1994):

$$\left|\hat{\boldsymbol{n}},\theta;p\right\rangle \equiv \frac{1}{\sqrt{p\,!}}\left(ua^\dagger + vb^\dagger - \theta f^\dagger\right)^p \left|0\right\rangle$$

Compute spin and superconducting correlators :

$$C_{\rm spin}(n) = \left\langle\, \boldsymbol{L}(j)\cdot\boldsymbol{L}(j+n)\,\right\rangle \quad , \quad C_{\rm SS}(n) = \left\langle \epsilon^{\mu\nu}\, b_{j,\mu}\, b_{j+n,\nu}\, f_j^\dagger\, f_{j+n}^\dagger \right\rangle$$



Conclusions

- VBS states a fertile playground for models of quantum magnetism

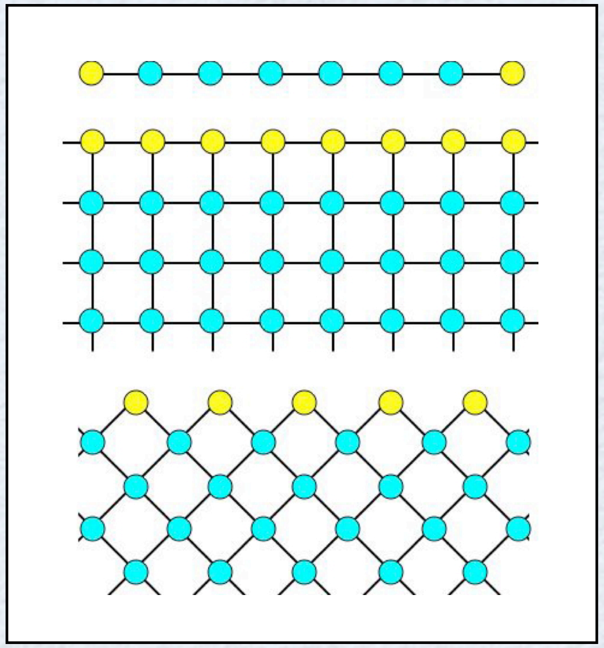
- VBS states are simplest example of matrix/tensor product states

- Equal time quantum correlations equivalent to finite temperature correlations of an associated classical model on the same lattice

- S = 2 VBS on diamond lattice is a quantum paramagnet, but S = 3 VBS on cubic lattice already possesses Néel order

- Abundant generalizations exist:

C-breaking SU(2n) states	Marston et al., 1991
SO(n) VBS chains	Tu et al., 2008
SU(4) plaquette ladders	Chen et al., 2005



- Fractionalization of representation at an edge

- Several analogies to FQHE physics