Quantum Simulation
of
Strongly Correlated Quantum Systems

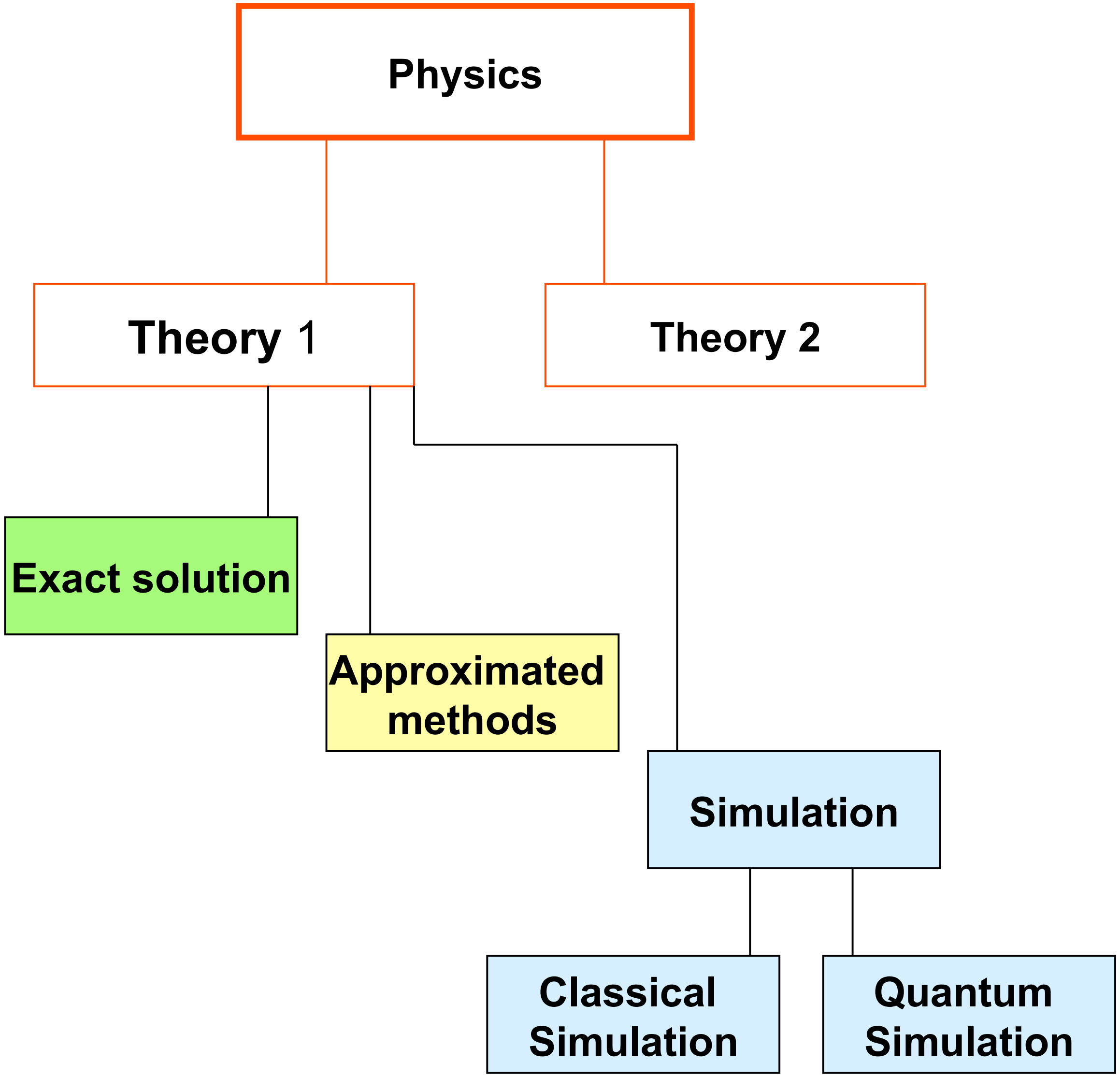
José I. Latorre

Evora, November, 2008

UB group

N. Barberán, S. Iblisdir, A. Celi

A. Riera, D. Dagnino, V. Picò, O. Boada


Physics
Theory 1
Theory 2
Exact solution
Approximated methods
Simulation
Classical Simulation
Quantum Simulation

Device

Classical Computer

(Classical algorithms)

Quantum Computer

(Quantum algorithms)

Problem

Classical problems

- Parity check
- Search
- Factorization

Quantum problems

- Ising model
- Superconductivity
- **TNS**

Exp gain

Classical problems

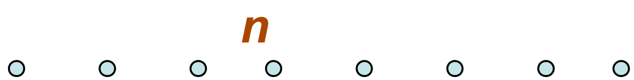
- Parity check
- Search
- Factorization

Quantum problems

- Ising model
- Superconductivity

Classical simulation of quantum dynamics

- **Exponential growth of Hilbert space**



$$|\Psi\rangle = \sum_{i_1=1}^{d} \dots \sum_{i_n=1}^{d} c_{i_1 \dots i_n} \, | i_1 \dots i_n \rangle$$

Classical representation requires ***d*** complex coefficients

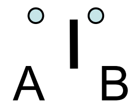
- **A random state carries maximum entropy**

$$\rho_L = Tr_{(n-L)} |\psi\rangle\langle\psi|$$

$$S(\rho_L) \equiv -Tr\left(\rho_L \log \rho_L\right) \approx L \log d$$

Efficient description for slightly entangled states

Schmidt decomposition



$$H = H_A \otimes H_B$$

$$|\Psi\rangle_{AB} = \sum_{i_1=1}^{\dim H_A} \sum_{i_2=1}^{\dim H_B} c_{i_1 i_2} \, |i_1\rangle_A \, |i_2\rangle_B$$

$$c_{i_1 i_2} = U_{i_1 k} \sqrt{p_k} V_{k i_2}^{+}$$

$$|\Psi\rangle_{AB} = \sum_{k=1}^{\chi} \sqrt{p_k} \, |\xi_k\rangle_A \, |\zeta_k\rangle_B$$

$$c_{i_1 i_2} = \sum_{k=1}^{\chi} \Gamma_k^{[1] i_1} \lambda_k \Gamma_k^{[2] i_2}$$

Retain eigenvalues and changes of basis

Vidal 03: Iterate this process

$$|\Psi\rangle = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} c_{i_1 \ldots i_n} |i_1 \ldots i_n\rangle$$

$$c_{i_1 \ldots i_n} = \sum_{\alpha_1 \ldots \alpha_{n-1}} \Gamma^{[1]i_1}_{\alpha_1} \lambda^{[1]}_{\alpha_1} \Gamma^{[2]i_2}_{\alpha_1\alpha_2} \lambda^{[2]}_{\alpha_2} \Gamma^{[3]i_3}_{\alpha_2\alpha_3} \ldots \Gamma^{[n]i_n}_{\alpha_{n-1}}$$

A product state iff $\alpha_i = 1$

$$\#\, parameters \approx nd\chi^2 << d^n$$

Slight entanglement iff $\chi \sim poly(n) << d$

- Representation is efficient
- Single qubit gates involve only local update
- Two-qubit gates reduces to local updating

} efficient simulation

Matrix Product States

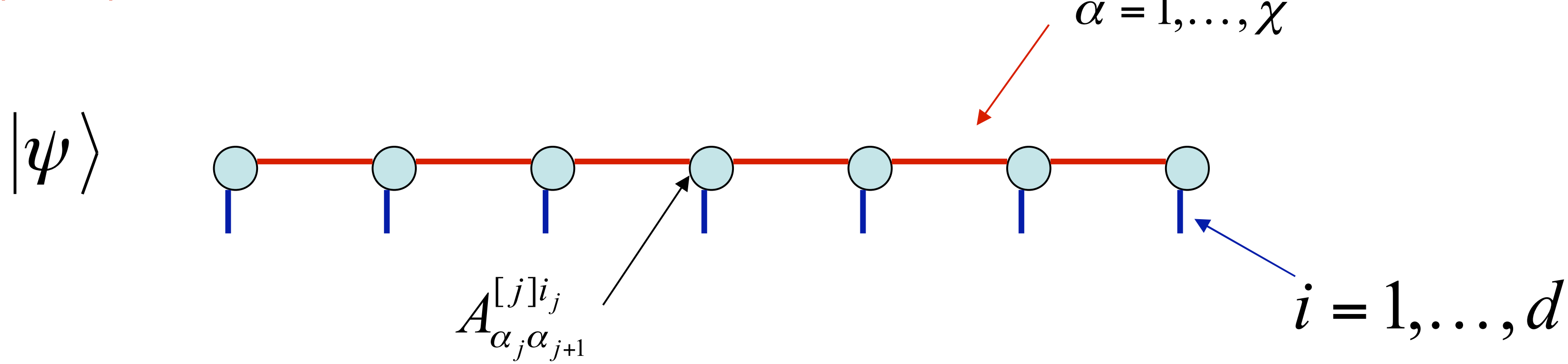
$$|\Psi\rangle = \sum_{i_1=1}^{d} \cdots \sum_{i_n=1}^{d} c_{i_1 \ldots i_n} \, | i_1 \ldots i_n \rangle$$

$$A = \Gamma\lambda$$

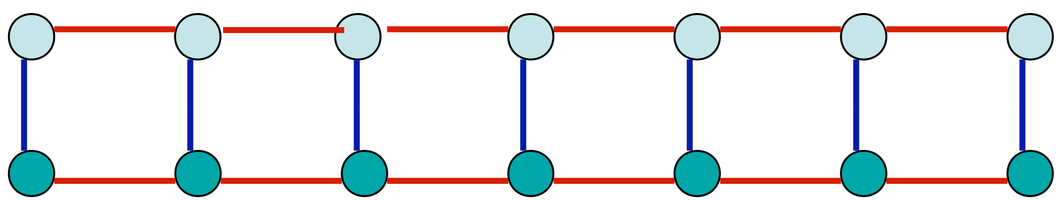
$$c_{i_1 \ldots i_n} = \sum_{\alpha_2 \ldots \alpha_n = 1,..\chi} A^{[1]i_1}_{1\alpha_2} A^{[2]i_2}_{\alpha_2\alpha_3} A^{[3]i_3}_{\alpha_3\alpha_4} \ldots . A^{[n]i_n}_{\alpha_n 1}$$

Approximate physical states with a finite χ MPS

Graphic representation of a MPS

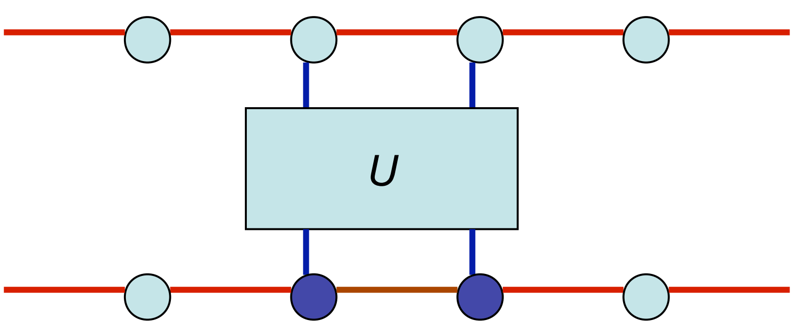


Efficient computation of scalar products



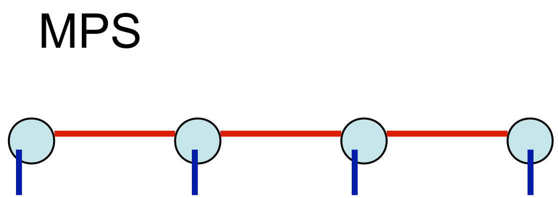
$nd\chi^3$ operations

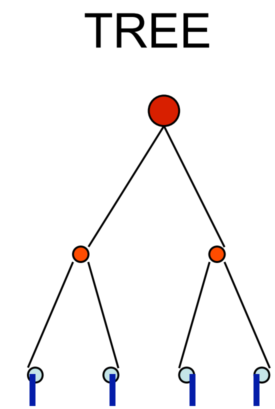
Action of local gates

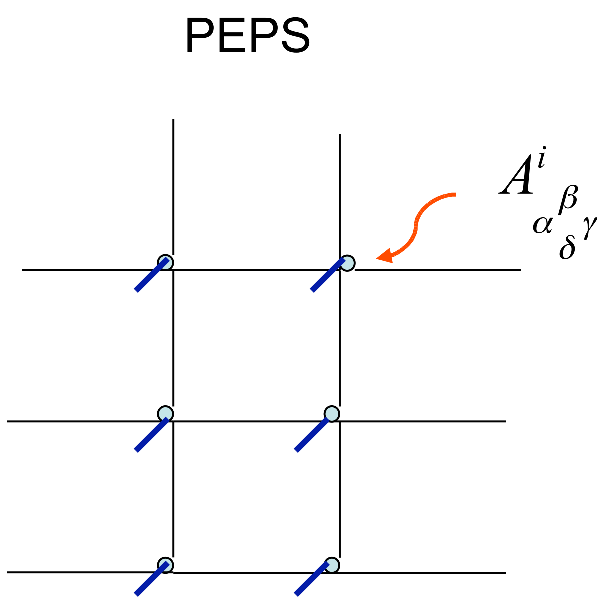


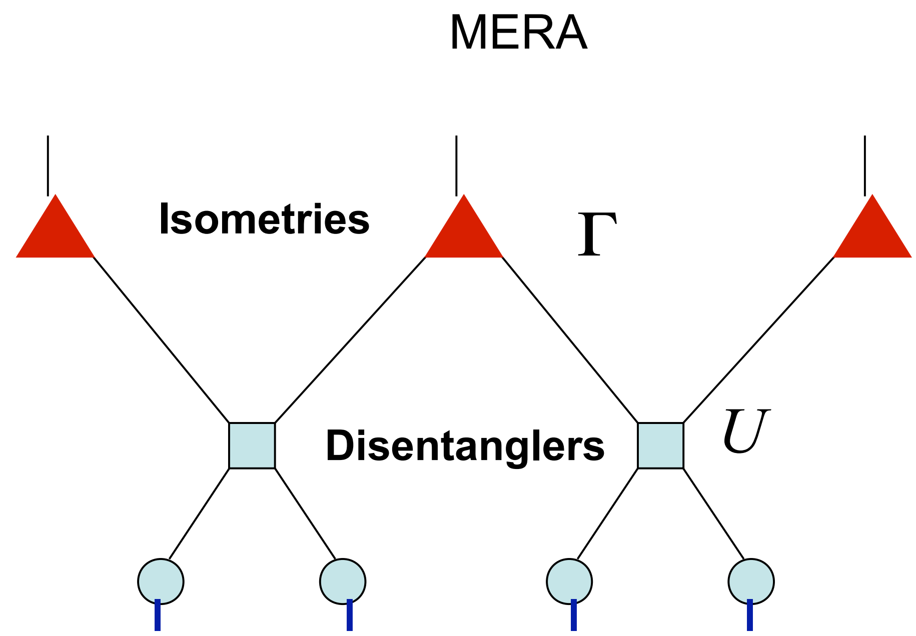
$$U^{kl}_{ij}\Gamma^{i}_{\alpha\beta}\lambda_{\beta}\Gamma^{j}_{\beta\gamma} = \tilde{\Gamma}^{k}_{\alpha\delta}\tilde{\lambda}_{\delta}\tilde{\Gamma}^{l}_{\delta\gamma}$$

Tensor Network representation of states

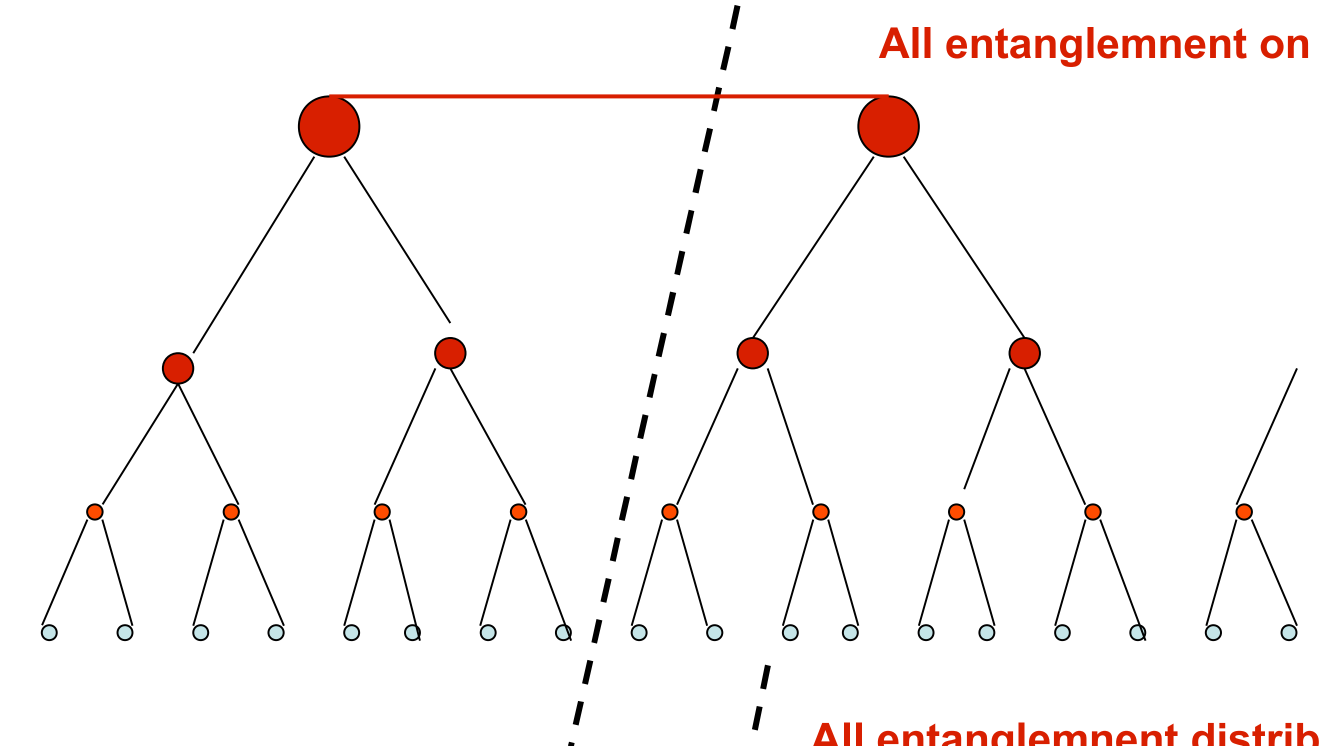




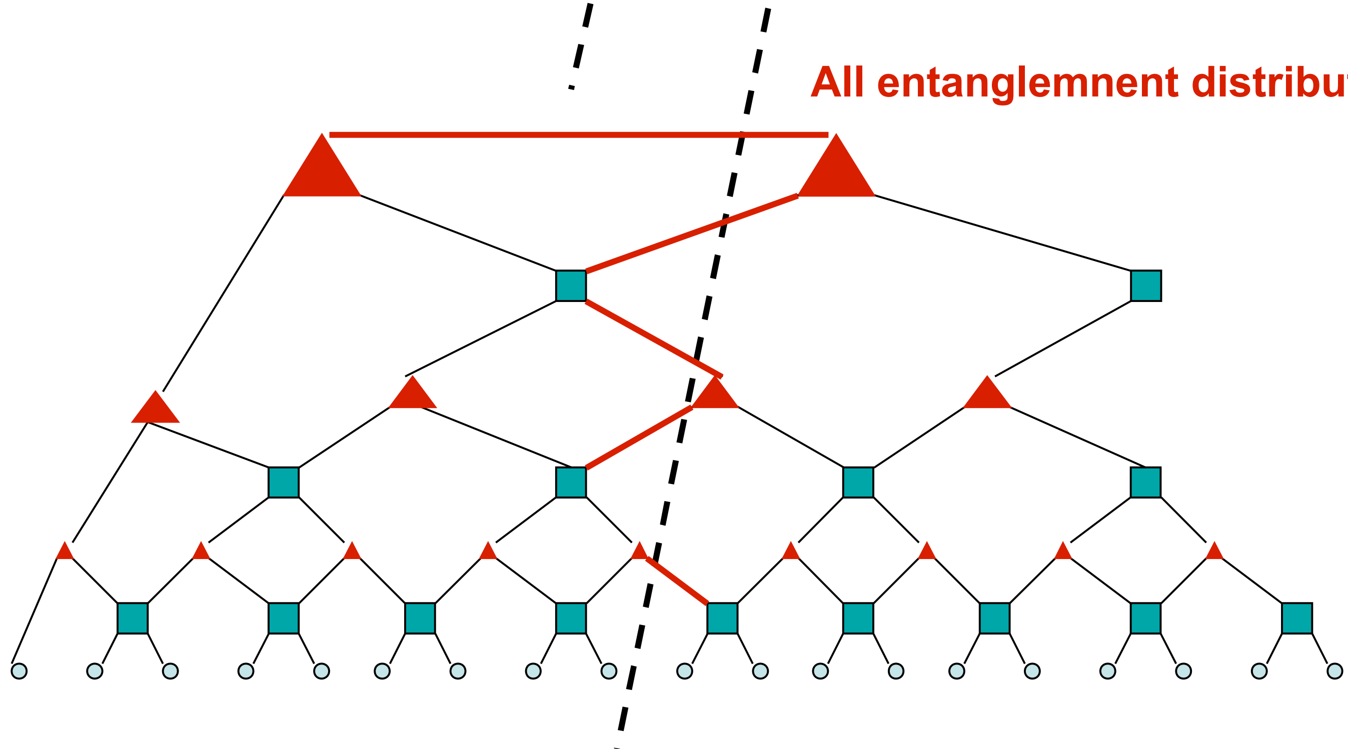




All entanglemnent on one line



All entanglemnent distributed on scales



Key idea:

All Tensor Networs are based on isometries

Isometries truncate the Hilbert space to a manageable size

Connectivity of the tensor captures entanglement between parts

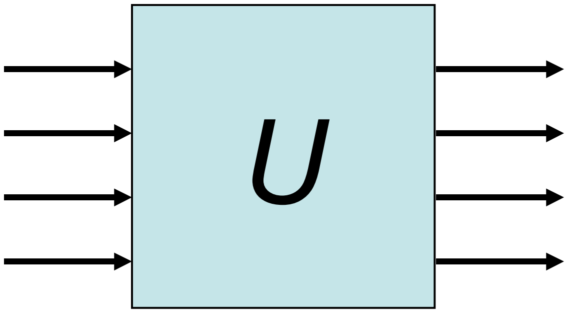
True quantum dynamics is based on unitary evolution
Quantum circuits are tensor networks made out of unitaries

Quantum Simulation

Frank Verstraete, Ignacio Cirac, JIL
arXiv:0804.1888

Problem: **simulate the Quantum Ising model with a Quantum computer**

$|\psi\rangle_{free}$



$|\psi_0\rangle_{ISING}$

$$|\psi_0\rangle_{ISING} = U|\psi\rangle_{free}$$

Previous efforts involved Trotter like approximations

$$\left|\psi_0\right\rangle_{ISING} = U\left|\psi\right\rangle_{free}$$

$$U(t,0) = V_n e^{-i\Delta t H_{aux}} V_{n-1} \ldots e^{-i\Delta t H_{aux}} V_1$$

Local unitaries

Auxiliary interaction

Can we find the **exact disentangler** of a realistic theory?

$$H_{ISING} = U H_{free} U^{+}$$

$$H_{free} = \sum_i \varepsilon_i \sigma_i^z$$

If so,

- We can prepare any state in a lab
  (fermions simulated on ion traps!)

- We can produce time evolution without time

$$e^{-itH_{ISING}} = U e^{-itH_{free}} U^{+}$$

- We can produce finite temperature at zero temperature

Thermal states $$e^{-\beta H_{ISING}} = U e^{-\beta H_{free}} U^{+}$$

$$H = \sum_i \sigma^x{}_i \sigma^x{}_{i+1} + \lambda \sum_i \sigma^z{}_i$$

$$\left|\psi\right\rangle_{ISING} = U(\lambda)\left|\psi\right\rangle_{free}$$

Follow: Jordan-Wigner + Fourier transform + Bogoliubov

Jordan-Wigner $\quad c_i = \prod_{j<i} \sigma_j^z \sigma_i^-$

$$\left|\psi\right\rangle = \sum A^{i_1 i_2 \ldots i_n} \left| i_1 i_2 \cdots i_n \right\rangle \qquad \left| i_j \right\rangle = \left\{ \left|\uparrow\right\rangle, \left|\downarrow\right\rangle \right\}$$

$$\left|\psi\right\rangle = \sum A^{i_1 i_2 \ldots i_n} (c_1^+)^{i_1} (c_2^+)^{i_2} \ldots (c_n^+)^{i_n} \left| 0_1 0_2 \ldots 0_n \right\rangle$$

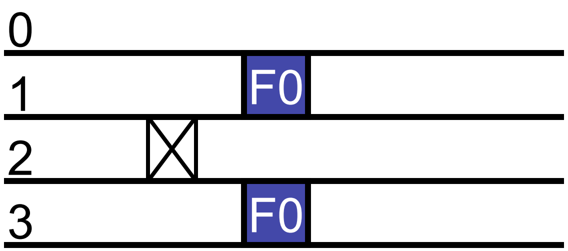
No change in *A*, no need for *U*

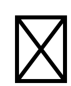
from now on modes are fermionic

Fast Fourier Transform

$$b_k = \sum_{j=0}^{n-1} c_j e^{-i2\pi \frac{kj}{n}}$$

$$b_k = \left(c_0 + c_2 e^{-i2\pi \frac{2k}{n}} \right) + e^{-i2\pi \frac{k}{n}} \left(c_1 + c_3 e^{-i2\pi \frac{2k}{n}} \right)$$





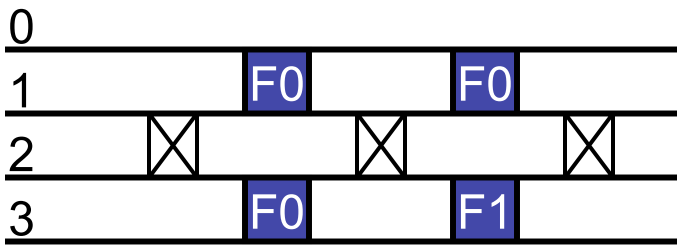
$$= U(fSWAP) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$c_0' = c_0 + c_2 \qquad c_1' = c_1 + c_3$$

$$c_2' = c_0 - c_2 \qquad c_3' = c_1 - c_3$$

$$b_0 = c_0' + c_1' \qquad b_1 = c_2' + e^{-i\pi\frac{1}{2}} c_3'$$

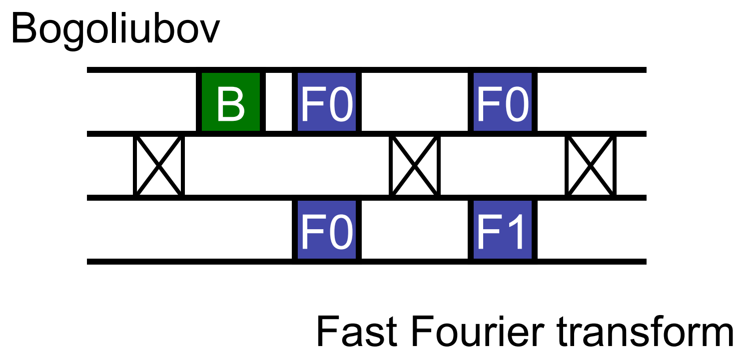
$$b_2 = c_0' - c_1' \qquad b_3 = c_2' - e^{-i\pi\frac{1}{2}} c_3'$$



$$|\psi\rangle = A'^{ij} (c'_1)^i (c'_2)^j |00\rangle = A'^{ij} \left(\frac{c_1 + e^{-i\pi\alpha} c_2}{\sqrt{2}} \right)^i \left(\frac{c_1 - e^{-i\pi\alpha} c_2}{\sqrt{2}} \right)^j |00\rangle$$

$$F\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{e^{-i\pi\alpha}}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{e^{-i\pi\alpha}}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & -e^{-i\pi\alpha} \end{pmatrix}$$

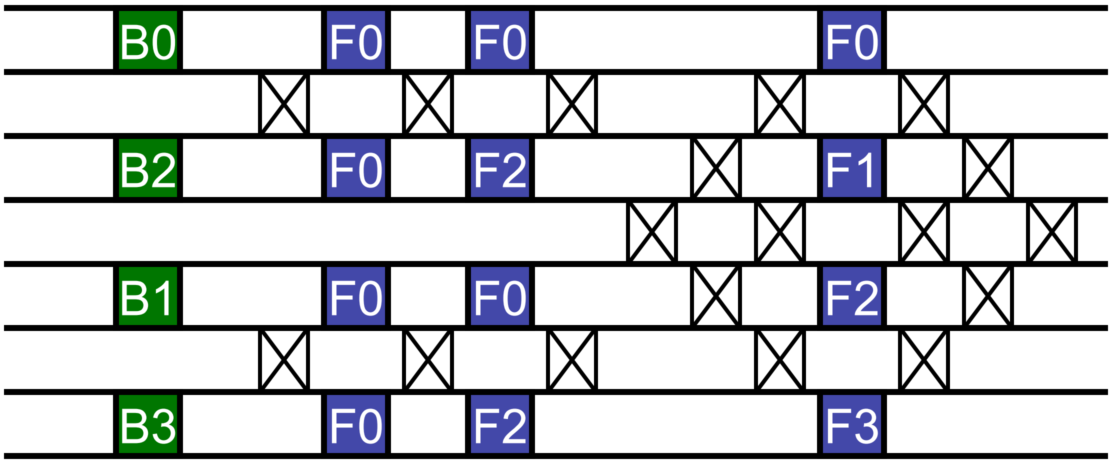
Quantum circuit for 4-qubit



$$\vartheta(\lambda) = ArcTan\left(\lambda - \sqrt{1+\lambda^2}\right)$$

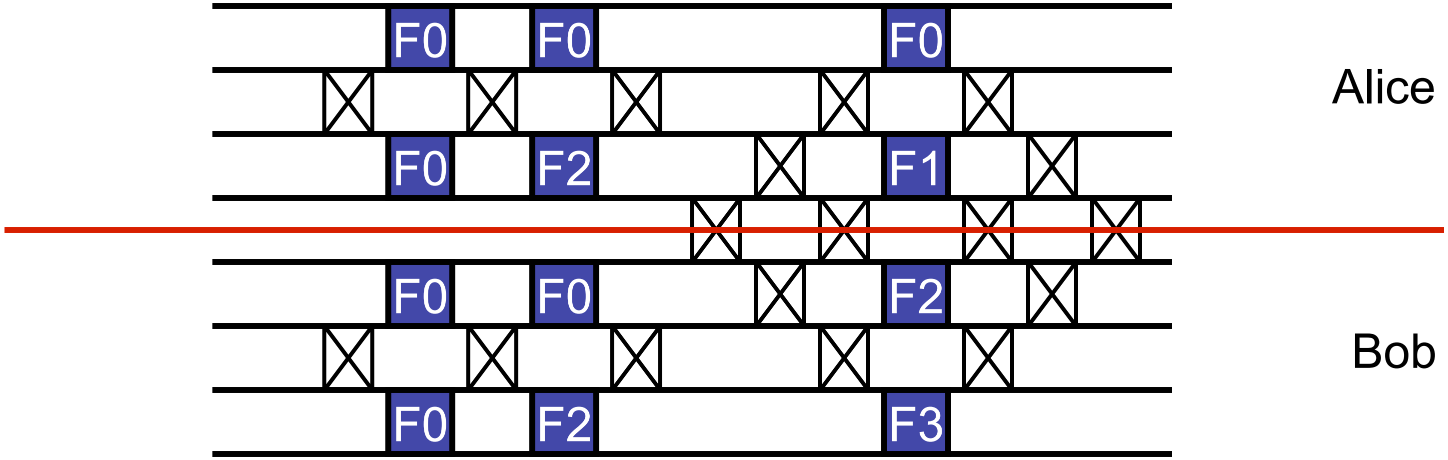
$$U(B) = \begin{pmatrix} \cos(\vartheta(\lambda)) & 0 & 0 & i\sin(\vartheta(\lambda)) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i\sin(\vartheta(\lambda)) & 0 & 0 & \cos(\vartheta(\lambda)) \end{pmatrix}$$

XY quantum circuit



$B(\lambda, \gamma)$ F are fixed

- **Depth n log(n)**
- **Periodicity is provided by the circuit**
- **Gates are local and not fined-tuned**

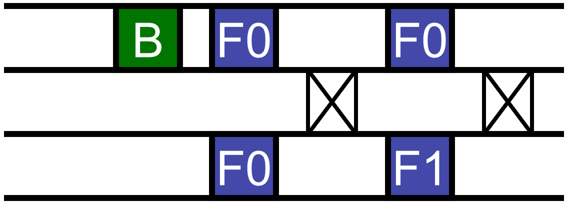


The circuit carries the minimum amount of gates connecting A and B to generate maximal entanglement

$$\left|\Psi(t)\right\rangle = e^{-itH_{\text{Ising}}}\left|\Psi(0)\right\rangle$$

Max entangled random

Proposal



7 gates build up any Ising state
External field is a parameter in B
Create fermions from ions, atoms, flux qubits,…

2-, 3- and 4-body correlators could be measured
Analysis of approximate quantum phase transition

Within experimental reach?

Towards Laughlin wave function

$$\Psi \approx \Pi_{ij}(z_i - z_j)^m e^{-\frac{1}{2}\sum_i z_i^2}$$

Extremely hard to simulate

m=1 $$\Psi \approx \varepsilon^{i_1 \dots i_n} \phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)$$

MPS representation
Iblisdir, JIL, Orús

$$\Psi \approx tr(\gamma^{i_1} \dots \gamma^{i_n} \gamma_5) \phi_{i_1}(z_1)\dots\phi_{i_n}(z_n)$$

Matrix product state with $\dim\gamma = 2^{\frac{n}{2}}$ $S \approx n$

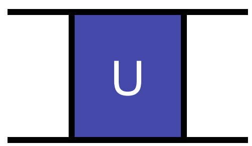
Simplest case:

Make a circuit for $$\Psi \approx \varepsilon^{i_1 i_2} \phi_{i_1}(z_1)\phi_{i_2}(z_2)$$

$\uparrow$

$|0\rangle, |1\rangle$

$|1\rangle$
$|0\rangle$

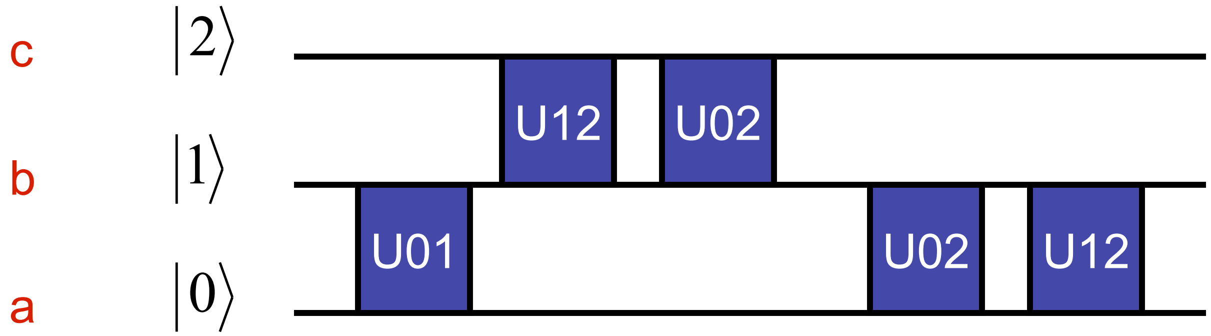


$$|1\rangle|0\rangle - |0\rangle|1\rangle$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Make a circuit for $\Psi \approx \varepsilon^{i_1 i_2 i_3} \phi_{i_1}(z_1)\phi_{i_2}(z_2)\phi_{i_3}(z_3)$

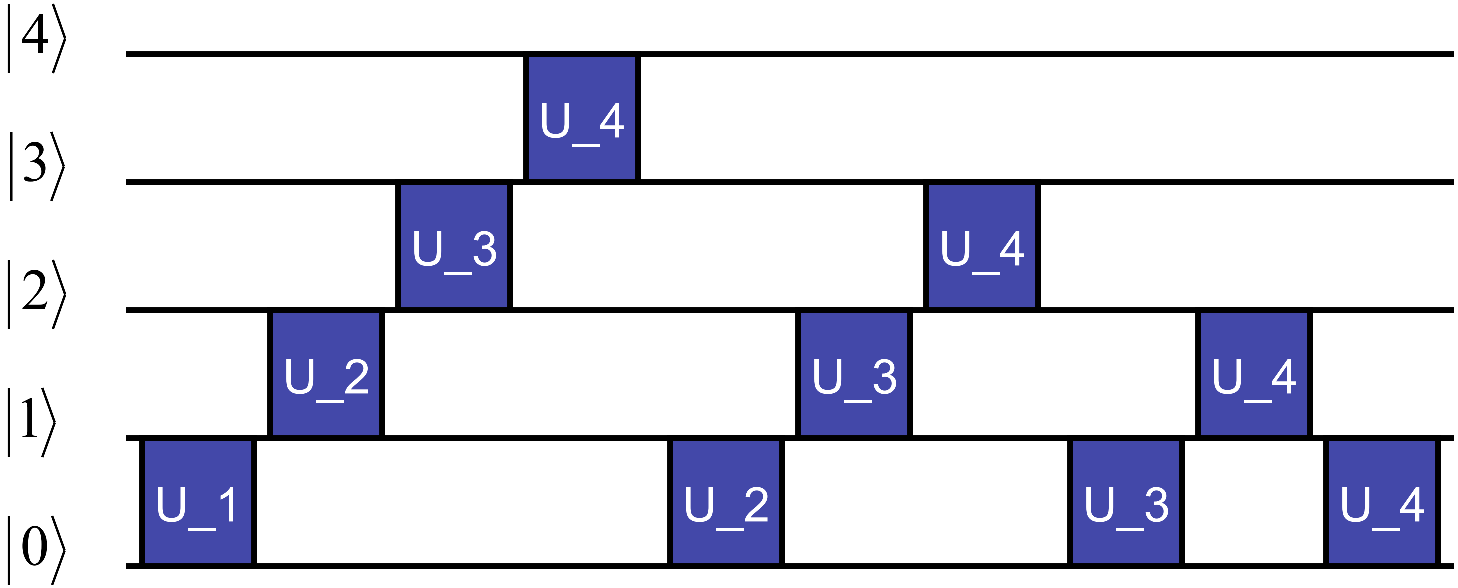
$$|012\rangle \rightarrow \frac{1}{\sqrt{6}}|012 - 021 + 120 - 102 + 201 - 210\rangle$$



$$U01^{ab} = U02^{ab} = U12^{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U12^{bc} = U02^{bc} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{\sqrt{2}}{\sqrt{3}} & 0 \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Arbitrary n

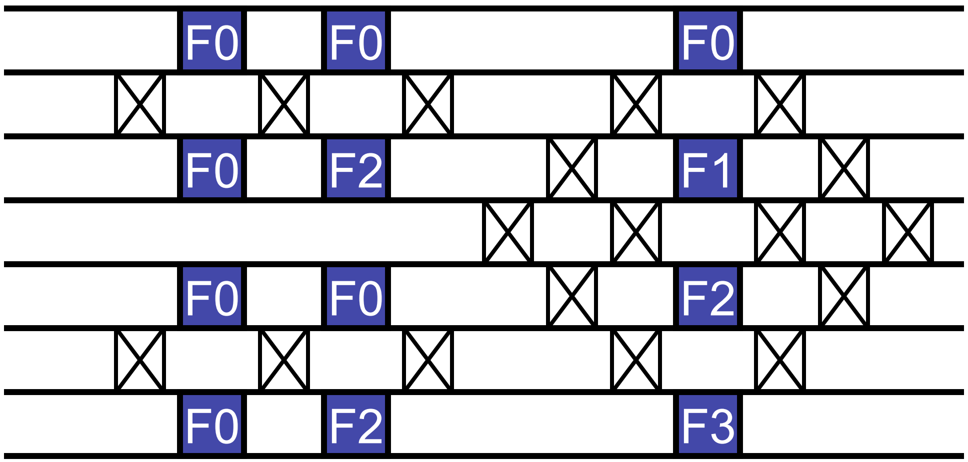


This can be further reduced to qubits with polynomial effort

Number of gates: $\frac{1}{3}n(n-1)(n-\frac{1}{2})$

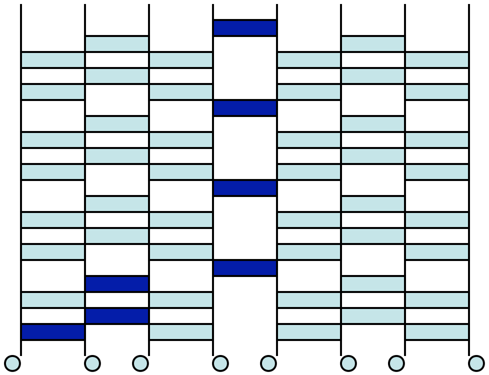
m=2,3,.. ?

Translational Invariance
+
Possibility of maximal entanglement



What’s needed for reparametrization invariance?

TNS>MERA

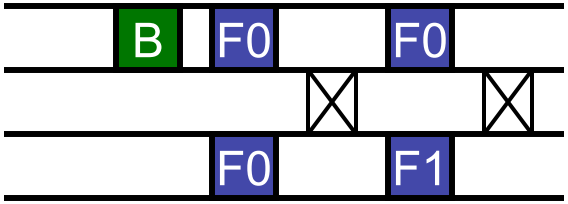


n entanglers at level n

MERA: log(n) entanglers

Conclusion

Exact circuits for quantum dynamics are within reach



XY exact circuit is polynomial, not fine tuned, optimal
It incorporates translational invariance
It is a complete disentangler

Exact circuits are the ultimate tensor networks made with unitaries