Workshop on Correlations, criticality, and coherence in quantum systems: AdS/CFT DUALITY

I. THE QCD STRING

A. QCD

B. STRINGS

II. THE ADS/CFT DUALITY

A. $\mathcal{N}=4$ SYM

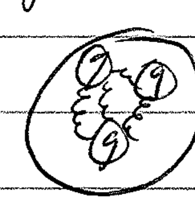
B. $AdS_5 \times S_5$

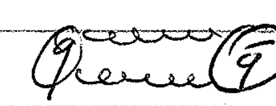
C. ADS/CFT CONJECTURE

I. THE QCD STRING

A. QCD

(i) Strong interactions described by non-Abelian gauge theory. Hadrons (baryons and mesons) are made of quarks, which are held together by exchange of gluons.





Lagrangian is ($m_q = 0$)

$$\frac{1}{g_{YM}^2}\int d^4x \left(-\frac{1}{2}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + i\,\bar{\Psi}_a \not{D}_{ab} \Psi_b \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$

- Gluon field is matrix valued, $A_\mu = A_\mu^A T^A$, $A = 1, \ldots, N^2-1$ (adjoint rep of SU(N))
  $\hookrightarrow$ generators of SU(N)
  $[T^A, T^B] = i f^{ABC} T^C$
  $\mathrm{tr}(T^A T^B) = \delta^{AB}/2$

$\boxed{\text{QCD } N=3}$

- Quarks in fundamental of SU(N), Ψ_a, $a = 1, \ldots, N$
  $$\not{D}_{ab}\Psi_b = \gamma^\mu\left(\partial_\mu \delta_{ab} - i A_\mu^c t^c_{ab}\right)\Psi_b$$
  $\rightarrow$ SU(N) generators in fundamental repr. (Gell Mann matrices)

(ii) Theory invariant under local gauge symmetry that generalizes the standard U(1) gauge symmetry of EMG

EMG: $\psi \to e^{i\theta(x)}\psi$ leaves action invariant

$$A_\mu \to A_\mu + i\partial_\mu \theta(x)$$

SU(N): $\psi_a \to e^{i t^c_{ab}\theta^c(x)}\psi_b \equiv \Omega_{ab}(x)\psi_b$

$$A_\mu \to \Omega(x) A_\mu \Omega^{-1}(x) + i(\partial_\mu \Omega(x))\Omega^{-1}(x)$$

(iii) QFT ~~[illegible]~~ and experiment tells us important dynamical information

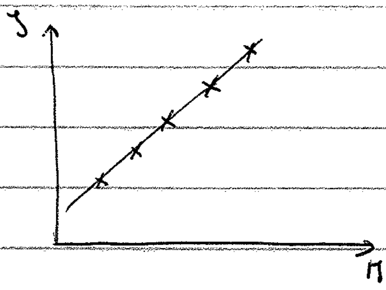
- Coupling constant runs with $\beta < 0$ (theory weakly coupled at high energies allowing for spectacular experimental tests at accelerators)

- At lower energies ~ 1 GeV theory is strongly coupled. This is a big dificulty because standard diagramatic Feynman techniques are helpless. We can't even predict the hadron mass spectrum!

- Confinement: we only see colour singlet bound states.

TWO IMPORTANT FACTS

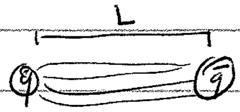
FACT 1 Meson spectrum aligned on Regge trajectories



$$J = \alpha(0) + \alpha' M^2 \quad , \alpha' \sim 1\,\text{GeV}^{-2}$$

$$[\alpha'] = \frac{E\cdot L}{E^2} = \frac{L}{E} \quad \text{inverse of tension}$$

- Is QCD a string theory?



Flux lines of gluon field forming a string $V \sim \frac{L}{\alpha'}$

FACT 2 Large N expansion ('t Hooft)

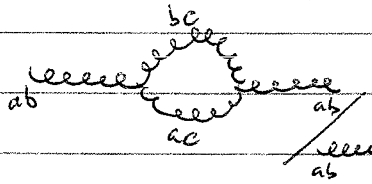
For large N theory simplifies keeping important dynamical features (then consider $\frac{1}{N}$ expansion)

$A_\mu \equiv N^2-1$ gluons

$\psi_a \equiv N$ colours ($\times N_f$ from flavour)

$N \to \infty$ expects that gluons dominate

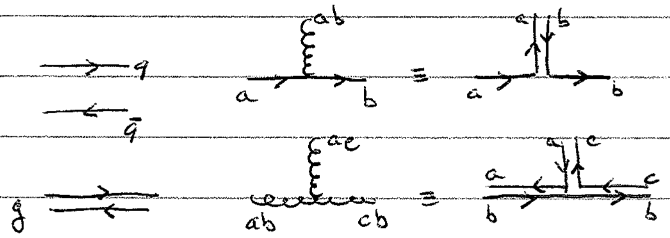
For example mass renormalization still present for $N \to \infty$



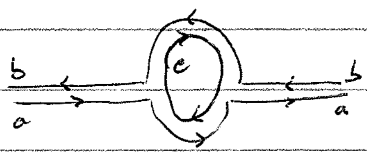
$$= (g^2_{YM})^3 \frac{N}{(g^2_{YM})^2} \sim g^2_{YM} N \equiv \lambda, \quad \left.\begin{matrix} N \to \infty \\ g_{YM} \to 0 \end{matrix}\right\} \lambda \text{ fixed}$$

't Hooft coupling

- Double line notation

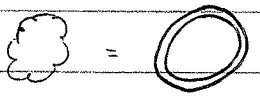


Previous mass renorm.

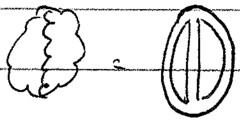


closed line ⇒ factor of N

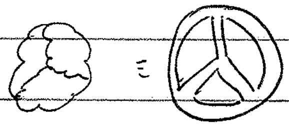
- Important fact is that Feynman diagrams organize in a doubles series expansion of $\frac{1}{N}$ and λ. Consider gluon vacuum diagrams



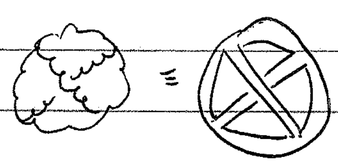
$\sim N^2$



$$\sim (g^2_{YM})^3 \left(\frac{1}{g^2_{YM}}\right)^2 N^3 = g^2_{YM} N^3 = N^2 \lambda$$

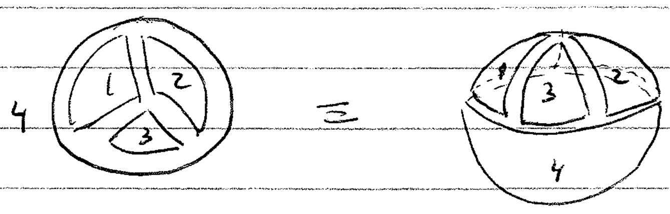


$$\sim (g^2_{YM})^6 \left(\frac{1}{g^2_{YM}}\right)^4 N^4 = g^4_{YM} N^4 = N^2 \lambda^2$$



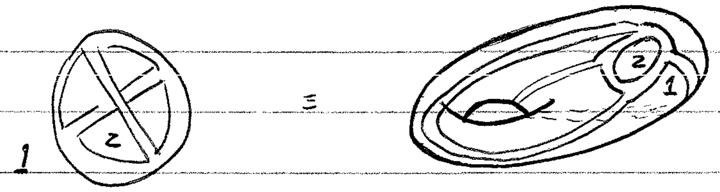
$$\sim (g^2_{YM})^6 \left(\frac{1}{g^2_{YM}}\right)^4 N^2 = g^4_{YM} N^2 = N^2 \frac{\lambda^2}{N^2}$$

Each diagram can be associated to a Riemann surface, with each line a boundary of a face and then gluing faces



Sphere Euler number $\chi = 2$

$\chi = 2 - 2g$ ($g = 0$ is genus) # handles



Torus $\chi = 0$ ($g = 1$)

faces F
edges E
vertices V

$$N^F (g_{YM}^2)^{E-V} = (N g_{YM}^2)^F (g_{YM}^2)^{E-V-F} = \lambda^F (g_{YM}^2)^{-\chi}$$
$$= \lambda^{F-\chi} \left(\frac{1}{N}\right)^{-\chi}$$

Expansion takes form $$\sum_{g=0}^{\infty} N^{\chi} \sum_{n=0}^{\infty} c_{g,n} \lambda^n , \quad \chi = 2 - 2g, \quad n = F - \chi$$

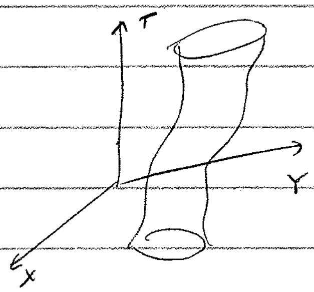
This fact is important because this expansion looks like a string theory loop expansion with $g_s \sim \frac{1}{N}$

B. STRINGS

(i) Classical action of a string moving in arbitrary curved space is (Polyakov)

$$S = -\frac{1}{4\pi\alpha'} \int d\tau\, d\sigma \sqrt{-\gamma}\, \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X)$$

Tension $= \frac{1}{2\pi\alpha'}$ $\quad [\alpha'] = \frac{L}{E} = L^2$



World-sheet coordinates $\sigma^a = (\tau, \sigma)$

World-sheet fields $X^\mu = X^\mu(\tau, \sigma)$

$\gamma_{ab} = \gamma_{ab}(\tau, \sigma)$

This is a two-dimensional field theory

Varying w.r.t metric γ_{ab} gives constraint

$$T_{ab} = -4\pi\alpha' \frac{1}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{ab}} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} - \frac{1}{2}\gamma_{ab}\gamma^{cd}\partial_c X^\mu \partial_d X^\nu g_{\mu\nu} = 0$$

and w.r.t. X's the eqns motion

$$\partial_a\left(\sqrt{-\gamma}\,\gamma^{ab}\partial_b X^\nu g_{\mu\nu}(x)\right) - \frac{\sqrt{-\gamma}}{2}\gamma^{ab}\partial_a X^\alpha \partial_b X^\beta \partial_\mu g_{\alpha\beta}(x) = 0$$

(ii) To find solutions it is convenient to work in Conformal gauge. Using 2D reparametrization invariance 2D metric can always be written as

$$\gamma_{ab} = e^{\varphi}\eta_{ab} \quad \text{with} \quad \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Action becomes

$$S = -\frac{1}{4\pi\alpha'}\int d\tau d\sigma\, \eta^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(x)$$

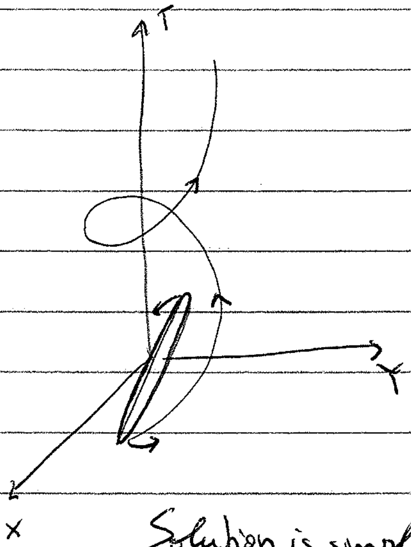
Eqns become: $T_{ab} = 0 \Rightarrow (\dot{X} \pm X')^\mu (\dot{X} \pm X')^\nu g_{\mu\nu} = 0 \Leftrightarrow (\dot{X} \pm X')^2 = 0$

$\dot{} \equiv \partial_\tau$, $' \equiv \partial_\sigma$

$$\partial_a\left(\eta^{ab}\partial_b X^\nu g_{\mu\nu}\right) - \frac{1}{2}\eta^{ab}\partial_a X^\alpha \partial_b X^\beta \partial_\mu g_{\alpha\beta} = 0$$

TWO IMPORTANT FACTS (to be compared with those of previous section)

FACT 1. RIGID ROTATING STRING



$$ds^2 = -dT^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2$$

$T = \tau$

$\phi = \omega\tau$

$\rho = \rho(\sigma)$

$$(\dot{X} \pm X')^2 = 0 \Rightarrow (\rho')^2 = (1-\omega^2\rho^2) \Rightarrow d\sigma = \frac{d\rho}{\sqrt{1-(\omega\rho)^2}}$$

Solution is simple: $\rho = \frac{\sin(\omega\sigma)}{\omega}$. Single folded string $\Rightarrow \sigma \in [0, \frac{2\pi}{\omega}]$

. Maximum ρ is $\rho_0 = \frac{1}{\omega}$

[check that ρ equation of motion is $\rho'' + \omega^2\rho = 0$]

In flat space $g_{\mu\nu} = \eta_{\mu\nu}$, string action is invariant under the global Poincaré transformations

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$$
$$\gamma_{ab} \to \gamma_{ab}$$

Under infinitesimal translations $X^\mu \to X^\mu + \epsilon^\mu$ we have $\delta \mathcal{L} = 0$. So Noether theorem says there is a conserved current ( )

$$P_\mu^a \epsilon^\mu = -\frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \epsilon^\mu \Rightarrow P^\mu_a = \frac{1}{2\pi\alpha'} \partial_a X^\mu$$

Conserved charge is the string space-time 4-momentum

$$P^\mu = \int d\sigma\, P^\mu_\tau = \frac{1}{2\pi\alpha'} \int d\sigma\, \partial_\tau X^\mu$$

Similarly, for infinitesimal Lorentz transformation $X^\mu \to \omega^\mu{}_\nu X^\nu$ with $\omega_{(\mu\nu)} = 0$ we have current

$$J^a_{\mu\nu} \omega^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \omega^\mu{}_\nu X^\nu \Rightarrow J^{\mu\nu}_a = \frac{1}{2\pi\alpha'}\left(X^\mu \partial_a X^\nu - X^\nu \partial_a X^\mu\right)$$

Conserved charge is the relativistic angular momentum

$$J^{\mu\nu} = \int d\sigma\, J^{\mu\nu}_\tau$$

Now look at previous spinning string

$$\begin{cases} x = \frac{\sin(\omega\sigma)}{\omega} \cos(\omega\tau) \\ y = \frac{\sin(\omega\sigma)}{\omega} \sin(\omega\tau) \end{cases}$$

$$\left(\sin^2(\omega\sigma) = \frac{1-\cos 2\omega\sigma}{2}\right)$$

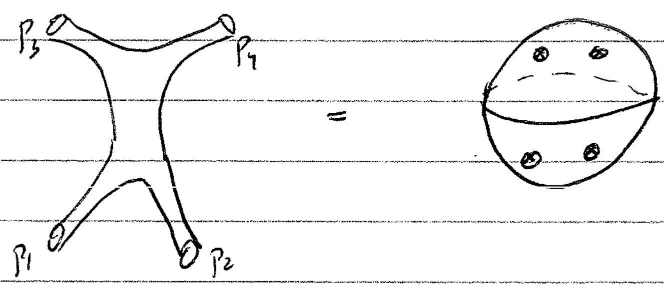
$$P^0 = E = \frac{1}{2\pi\alpha'} \frac{2\pi}{\omega} = \frac{1}{\alpha'\omega}$$

$$J^{xy} = J = \frac{1}{2\pi\alpha'} \int d\sigma \frac{\sin^2(\omega\sigma)}{\omega^2} \omega \big(\underbrace{\cos^2(\omega\tau) + \sin^2(\omega\tau)}_{1}\big) = \frac{1}{2\pi\alpha'} \frac{1}{\omega} \frac{1}{2} \frac{2\pi}{\omega} = \frac{1}{2\alpha'\omega^2}$$

This is the same Regge behaviour observed in QCD! $\quad J = \frac{\alpha'}{2} E^2$

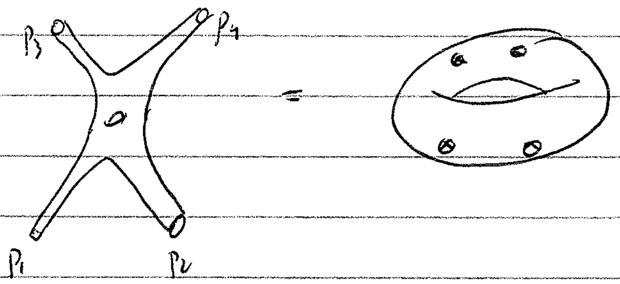
FACT 2

In string theory interactions are defined in terms of Feynman diagrams. For example 2-2 scattering



$$\chi = 2 - 2\underbrace{g}_{\text{handels}} - \underbrace{h}_{\text{holes}}$$

$\chi = -2$



$\chi = -4$

$$S(p_1, p_2, p_3, p_4) = \sum_{\text{topologies}} g_s^{-\chi} \int DX D\gamma \exp\left(-S[X,\gamma]\right) V_1 \cdot V_2 \cdots V_4$$

This is similar to the large N expansion $g_s \sim \frac{1}{N}$

CONCLUSION: Both facts 1 and facts 2 call for a string description of QCD

ADS/CFT IS A PRECISE FORMULATION OF THESE OLD IDEAS

II. THE ADS/CFT DUALITY

A. $\mathcal{N}=4$ SYM

(i) We shall consider a particular SU(N) gauge theory with all fields in the adjoint

$$A_\mu = A_\mu^A T^A, \quad \phi_i = \phi_i^A T^A \ (i=1,\dots,6), \quad \lambda_I = \lambda_I^A T^A \ (I=1,\dots 4)$$

Bosonic ~~[illegible]~~ action is

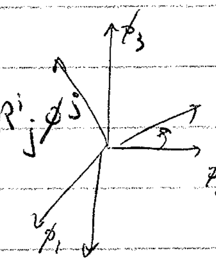
$$\frac{1}{g_{YM}^2}\int d^4x \,\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - D_\mu \phi_i D^\mu \phi^i + \frac{1}{2}[\phi_i,\phi_j][\phi^i,\phi^j]\right)$$

with $D_\mu = \partial_\mu - i[A_\mu, \ ]$ and $\phi^i = \delta^{ij}\phi_j$.

$Z(x) = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$
2-pt function is
$$\langle Z^A(x)\bar{Z}^B(y)\rangle = \frac{g_{YM}^2}{4\pi^2}\frac{\delta^{AB}}{(x-y)^2}$$

(ii) This theory has a lot of global symmetries

- Can rotate the 6 scalars with SO(6) transformations $\phi'^i = R^i{}_j\phi^j$ (those that leave δ_{ij} invariant). Gauge field A_μ is singlet



group SO(2,4)
- Lorentz transformations $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ (the SO(1,3) group)
  Gauge field A_μ is a vector and ϕ_i singlet.
- Translations $x^\mu \to x^\mu + a^\mu$
- Conformal transformations $x^\mu \to \lambda x^\mu$
  $$x^\mu \to \frac{x^\mu a^2 - a^\mu x^2}{(a-x)^2} \quad \text{(special conformal)}$$

This $SO(2,4)\times SO(6)$ symmetry also holds at quantum level, i.e. it is exact (fermions & SUSY)

iii) Conformal symmetry $x^\mu \to \lambda x^\mu$ means theory has no prefered scale. How can we characterize the spectrum?

Generally for a field or composite field operator

$$\Psi(\lambda x) = \lambda^{-\Delta}\Psi(x) \quad , \quad \Delta \text{ is the (mass) dimension}$$

(For example $[A_\mu] = [\phi_i] = L^{-1} = M^1$)

Consider 2-pt function of single trace operator $\mathcal{O}(x) = \mathrm{tr}(Z^2(x))$, $Z(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x))$
($\mathcal{O}$ is analogue of meson operator, but with scalar quarks in adjoint)

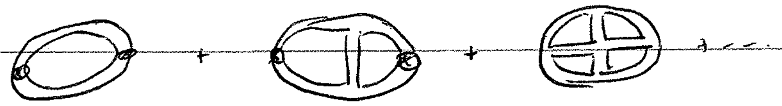
Can compute $\langle \mathcal{O}(x)\,\mathcal{O}^\dagger(y)\rangle = \langle \mathrm{tr}(Z(x)\,Z(x))\ \mathrm{tr}(\bar{Z}(y)\,\bar{Z}(y))\rangle$
from Wick contractions

$\mathcal{O}(x)$  $\mathcal{O}(y)$

$$= \left(\frac{g^2_{YM}}{4\pi^2}\right)^2 \frac{N^2-1}{(x-y)^{2\Delta}} \quad , \ \Delta \text{ is dimension in this case } \Delta = 2$$

(From Lorentz invariance, translation inv. and scaling property can not write anything else)

Just like in conventional QFT poles in 2-point function give information about spectrum, in CFT we can read from 2-point function the dimension of field operators. The quantum corrections give then the anomalous dimension

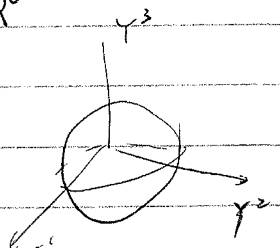


$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle \sim \frac{1}{(x-y)^{2(\Delta+\gamma)}}$$

γ ← quantum piece
(just like mass renormalization in QFT)

B. $AdS_5 \times S^5$

(i) A 5-sphere S^5 can be constructed from its embedding space $\mathbb{R}^6$



$$Y^2 = (Y_1)^2 + (Y_2)^2 + \cdots + (Y_6)^2 = R^2$$

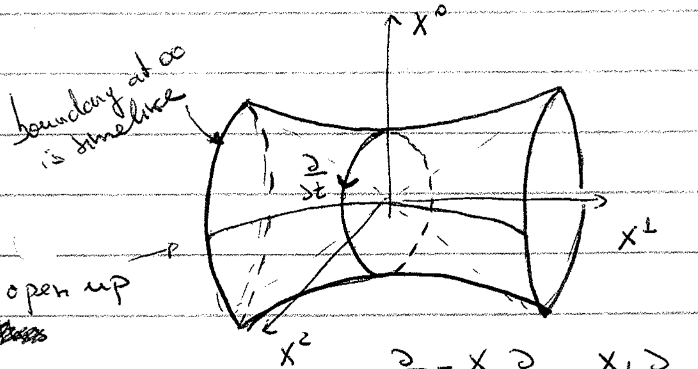
This is a constant curvature space with SO(6) rotational symmetry inherited from rotations in embedding space, with generators $J_{ij} = Y_i \frac{\partial}{\partial Y^j} - Y_j \frac{\partial}{\partial Y^i}$.

(ii) The AdS_d space is defined as the surface in $\mathbb{R}^{2,d-1}$ $(d=5)$

$$X^2 = -(X^0)^2 + (X^1)^2 + \cdots + (X^4)^2 - (X^d)^2 = -R^2 \qquad (+)$$

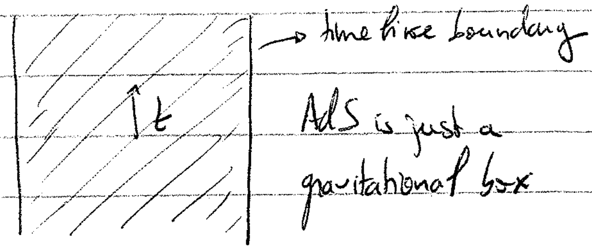
This pseudo-sphere preserves $SO(2,d-1)$ Lorentz symmetry of embedding space with generators $J_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A}$.

AdS_2 : $X^2 = -(X^0)^2 + (X^1)^2 - (X^2)^2$



$$\frac{\partial}{\partial t} = X_0 \frac{\partial}{\partial X^d} - X_d \frac{\partial}{\partial X^0}$$

AdS is obtained by decompactifying this closed time-like circle.



AdS is just a gravitational box

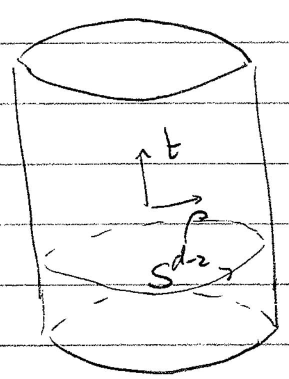
Now construct the AdS metric. Choose the following parametrization of the surface $(+)$

$$X^0 = R\,\mathrm{ch}\rho \cos t$$
$$X^d = R\,\mathrm{ch}\rho \sin t$$
$$X^i = R\,\mathrm{sh}\rho\, n^i$$

, with $\hat{n}$ the unit vector on S^{d-2}

Metric is just

$$ds^2 = dx^A dx_A = R^2 \left(-\mathrm{ch}^2\rho\, dt^2 + d\rho^2 + \mathrm{sh}^2\rho\, d\Omega^2_{d-2} \right)$$



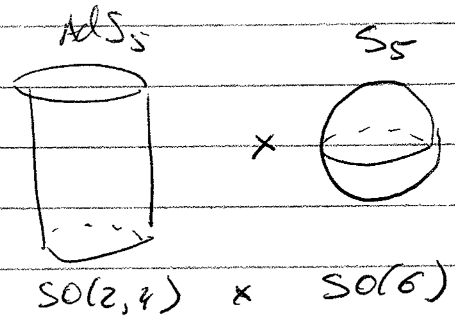
- AdS is like a cylinder. Boundary at $\rho \to \infty$ is conformal to $\mathbb{R} \times S^{d-2}$

$$ds^2 \xrightarrow[\rho\to\infty]{} \left(\frac{R e^{\rho}}{2}\right)^2 \left(-dt^2 + d\Omega_{d-2} \right)$$

- This geometry solves Einstein gravity with negative Λ

$$\int d^d x \sqrt{-g}\, (R - \Lambda)$$

C. AdS/CFT CONJECTURE



The space $AdS_5 \times S_5$ of radius R has the same global symmetries as $\mathcal{N}=4$ SYM

$$\boxed{\text{STRING THEORY ON } AdS_5 \times S_5 \equiv \mathcal{N}=4 \text{ SYM}}$$

(space-time isometries) $\quad SO(2,4) \times SO(6) \quad$ (conformal sym. & R-sym.)

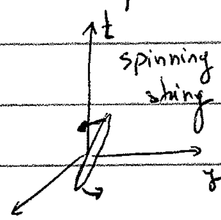
$\left(g_s \to 0 \text{ free strings} \right) \qquad g_s = \dfrac{R^4}{4\pi \alpha'^2} \dfrac{1}{N} \qquad \left(N \to \infty \text{ is planar theory} \right)$

$\left(\begin{array}{l} \lambda \gg 1 \text{ classical strings} \\ \text{corresponds to strong coupling SYM} \end{array} \right) \quad \lambda = \dfrac{R^4}{\alpha'^2} = g^2_{YM} N \quad \left(\begin{array}{l} \lambda \ll 1 \text{ is perturbative SYM} \\ \text{corresponds to highly quantum strings} \end{array} \right)$

Single string state $\equiv$ single trace operators

Examples: $g_{\mu\nu} \equiv T_{\mu\nu} = \mathrm{Tr}(F_{\mu\lambda} F_\nu{}^\lambda + \dots)$

$\phi \equiv \mathcal{L}_0 = \mathrm{Tr}(F^2 + \dots)$



$\equiv \mathcal{O} = \mathrm{tr}\left(\phi_i D_{(\mu_1} \cdots D_{\mu_s)} \phi_j \right)$

(energy of string state) $\quad E = \Delta \quad$ (dimension of dual operator)